Semantics and Domain theory

Exercises 5

1. Prove that, given the partial function $f : X \rightharpoonup Y$, the function $f_\perp : X_\perp \to Y_\perp$ is continuous. (Proposition 3.1.1 in Pitts' notes.) Here $f_\perp$ is defined by

$$f_\perp(d) := \begin{cases} f(d) & \text{if } d \in X \text{ and } f(d)\downarrow \\ \perp & \text{if } d \in X \text{ and } f(d)\uparrow \\ \perp & \text{if } d = \perp \end{cases}$$

Answer: ....
$X_\perp$ is a flat domain, so all chains in $X_\perp$ are eventually constant, so if $f : X_\perp \to Y_\perp$ is monotonic, it is continuous. (See Exercise 5 of week 3.) So, it suffices to show that $f_\perp : X_\perp \to Y_\perp$ as defined above is monotonic. Assume that $x_1 \sqsubseteq x_2$ in $X_\perp$. If $f_\perp(x_1) = \perp$, then $f_\perp(x_1) \sqsubseteq f_\perp(x_2)$ holds trivially, so we only have to consider the case when $f_\perp(x_1) \neq \perp$. Then $x_1 \in X$ and $f(x_1)\downarrow$. So, from $x_1 \sqsubseteq x_2$, we derive $x_2 = x_1$ and so $f(x_2) = f(x_1)$ and we are done.
End answer..

2. Prove that for D a domain and $F : (D \to D) \to (D \to D)$ and $g : D \to D$ continuous,

$$\mathrm{ev}(\mathrm{fix}(F), \mathrm{fix}(g)) = \sqcup_{k\geq 0} F^k(\perp')(g^k(\perp)),$$

where $\perp$ is in D and $\perp'$ is in $D \to D$ and ev is the evaluation function (of Proposition 3.3.1):

$$\mathrm{ev}(f, d) := f(d),$$

for $f : D \to D$ and $d : D$, so $\mathrm{ev} : (D \to D) \times D \to D$.

Answer: ....

$$\begin{aligned} \mathrm{ev}(\mathrm{fix}(F), \mathrm{fix}(g)) &= \mathrm{ev}(\sqcup_{i\in\mathbb{N}} F^i(\perp'), \sqcup_{j\in\mathbb{N}} g^j(\perp)) \\ &\stackrel{1}{=} \sqcup_{i\in\mathbb{N}} \sqcup_{j\in\mathbb{N}} \mathrm{ev}(F^i(\perp'), g^j(\perp)) \\ &\stackrel{2}{=} \sqcup_{i\in\mathbb{N}} \sqcup_{j\in\mathbb{N}} F^i(\perp')(g^j(\perp)) \\ &\stackrel{3}{=} \sqcup_{k\geq 0} F^k(\perp')(g^k(\perp)) \end{aligned}$$

where $\stackrel{1}{=}$ is by the continuity of ev (and thereby the continuity in each of the arguments separately), $\stackrel{2}{=}$ is the definition of ev and $\stackrel{3}{=}$ is by "diagonalising a double chain". It should be checked that the preconditions for applying the "diagonalising a double chain" Lemma are met, that is: for $i \leq n$ and $j \leq m$, we have $F^i(\perp')(g^j(\perp)) \sqsubseteq F^n(\perp')(g^m(\perp))$. This follows (using monotonicity) from

- if $j \leq m$, then $g^j(\perp) \sqsubseteq g^m(\perp)$,
- if $i \leq n$, then $F^i(\perp') \sqsubseteq F^n(\perp')$.

The proof of the latter two properties has been given in the proof of Tarksi's fixed-point theorem, where we have shown that $(f^i(\perp))_{i\in\mathbb{N}}$ is a chain.
End answer..

3. We define two variants of a functional $F : [\mathbb{N}_\perp \to \mathbb{N}_\perp] \to \mathbb{B}_\perp$ that, given a continuous function $f : \mathbb{N}_\perp \to \mathbb{N}_\perp$, checks if f has a fixed point in $\mathbb{N}$. (That is: an $n \in \mathbb{N}$ for which $f(n) = n$).

$$F_1(f) \;:=\; \begin{cases} \mathrm{tt} & \text{if } f \text{ is total and } \exists\, n \in \mathbb{N}(f(n) = n) \\ \mathrm{ff} & \text{if } f \text{ is total and } \forall\, n \in \mathbb{N}(f(n) \neq n) \\ \perp & \text{if } f \text{ is not total} \end{cases}$$

$$F_2(f) \;:=\; \begin{cases} \mathrm{tt} & \text{if } \exists\, n \in \mathbb{N}(f(n) = n \wedge \forall\, m < n(f(m) \neq m, \perp)) \\ \perp & \text{otherwise} \end{cases}$$

NB. "f is total" means that $\forall\, n \in \mathbb{N}(f(n) \neq \perp)$.

(a) Prove that both F_1 and F_2 are monotonic (in case this hasn't been checked in the lecture already).

Answer: ....

For F_1: suppose $f \sqsubseteq g$. We only have to consider the case when $F_1(f) \neq \perp$. If $F_1(f) = \mathrm{tt}$, then f is total. But then g is total and $f = g$, so $F_1(g) = \mathrm{tt}$. Similarly, if $F_1(f) = \mathrm{ff}$, then f is total and then g is total and $f = g$, so $F_1(g) = \mathrm{ff}$.

For F_2: suppose $f \sqsubseteq g$. We only have to consider the case when $F_2(f) \neq \perp$, so when $F_1(f) = \mathrm{tt}$. Then there is an $n \in \mathbb{N}$ with $f(n) = n$ and $\forall\, m < n(f(m) \neq m, \perp)$. As $f \sqsubseteq g$ we also have $g(n) = n$ and $\forall\, m < n(g(m) \neq m, \perp)$, so $F_2(g) = \mathrm{tt}$. Done.

End answer..

(b) Prove that one of the F_i does not preserve lubs (and thus is not continuous).

Answer: ....

F_1 does not preserve lubs. Consider the chain $(f_i)_{i \in \mathbb{N}}$ where

$$f_i(x) := \begin{cases} x & \text{if } x \leq i \\ \perp & \text{if } x = \perp \text{ or } x > i \end{cases}$$

Then $F_1(f_i) = \perp$, so $\sqcup_{i \in \mathbb{N}} F_1(f_i) = \perp$. On the other hand, $\sqcup_{i \in \mathbb{N}} f_i = \mathrm{Id}$ (the identity function on $\mathbb{N}_\perp$), so $F_1(\sqcup_{i \in \mathbb{N}} f_i) = \mathrm{tt}$.

End answer..

(c) Prove that one of the F_i preserves lubs (and thus is continuous).

Answer: ....

F_2 preserves lubs: we need to prove that $F_2(\sqcup_{i \in \mathbb{N}} f_i) \sqsubseteq \sqcup_{i \in \mathbb{N}} F_2(f_i)$ for all chains.

Let $(f_i)_{i \in \mathbb{N}}$ be a chain. Note that, if, for some $n \in \mathbb{N}$, $f_i(n) \in \mathbb{N}$, then $f_j(n) = f_i(n)$ for all $j \geq i$. We only have to consider the case when $F_2(\sqcup_{i \in \mathbb{N}} f_i) \neq \perp$, that is $F_2(\sqcup_{i \in \mathbb{N}} f_i) = \mathrm{tt}$. Then there is an $n \in \mathbb{N}$ with $(\sqcup_{i \in \mathbb{N}} f_i)(n) = n$ (a) and $\forall\, m < n((\sqcup_{i \in \mathbb{N}} f_i)(m) \neq m, \perp))$ (b).

By definition, (a) means $\sqcup_{i \in \mathbb{N}}(f_i(n)) = n$, from which we derive that $f_k(n) = n$ for some k, because chains in $\mathbb{N}_\perp$ are eventually constant. Similarly, from (b) we derive that for all $m < n$ there is a p_m such that $f_{p_m}(m) \neq m, \perp$. Take $M : \max\{k, p_0, \ldots, p_{n-1}\}$. Then $f_M(n) = n$ and $\forall\, m < M(f_M(m) \neq m, \perp$. So $F_2(f_M) = \mathrm{tt}$ and therefore $\sqcup_{i \in \mathbb{N}}(F_2 f_i) = \mathrm{tt}$.

End answer..

4. Let $P : D \to \mathbb{B}_\perp$ and $g : D \to D$ be continuous. Define $f : D \times D \to D \times D$ by
$$f(d_1, d_2) = \text{If}(P(d_1), (g(d_1), g(d_2)), (g(d_2), g(d_1))).$$
Show that for $\text{fix}(f) = (u_1, u_2)$, we have $u_1 = u_2$. (Use Scott induction.)

Answer: ....
We know that $\Delta := \{(d_1, d_2) \in D \times D \mid d_1 = d_2\}$ is an admissible subset of $D \times D$. (Treated in the lecture; see the notes by Pitts.) So we only have to show that Δ is closed under f: If (d_1, d_2) in Δ, then $f(d_1, d_2) \in \Delta$. (Then $\text{fix}(f) \in \Delta$ follows by Scott induction.)

So let $d \in D$. We prove that $f(d, d) \in \Delta$. There are two cases: $P(d) = \perp$, then $f(d, d) = \perp_{D\times D} = (\perp_D, \perp_D) \in \Delta$; $P(d) \neq \perp$, then $f(d, d) = \text{If}(P(d), (g(d), g(d)), (g(d), g(d))) = (g(d), g(d)) \in \Delta$. So done.

End answer..

5. (Exercise 3.4.2 of Pitts' notes): Let X and Y be sets and $X_\perp$ and $Y_\perp$ be the corresponding flat domains. Show that a function $f : X_\perp \to Y_\perp$ is continuous if and only if one of (a) or (b) holds:

   (a) f is strict, i.e. $f(\perp) = \perp$.
   (b) f is constant, i.e. $\forall\, x \in X(f(x) = f(\perp))$.

Answer: ....
We (again) use the following property: $X_\perp$ is a flat domain, so all chains in $X_\perp$ are eventually constant, so if $f : X_\perp \to Y_\perp$ is monotonic, it is continuous. (See Exercise 5 of week 3.) So it suffices to prove that a function $f : X_\perp \to Y_\perp$ is monotonic if and only if one of (a) or (b) holds.

For the if-and-only-if we need prove two directions:

If $f : X_\perp \to Y_\perp$ satisfies (b) it is constant and thus monotonic. If $f : X_\perp \to Y_\perp$ satisfies (a) and $x, x' \in X_\perp$ with $x \sqsubseteq x'$, there is only one non-trivial situation (where $x \neq x'$), which is when $x = \perp$ and $x' \in X$. Then $f(\perp) = \perp \sqsubseteq f(x')$, so done: f is monotonic.

The other way around: assume $f : X_\perp \to Y_\perp$ is monotonic. We have to show that, if $f(\perp) \neq \perp$, then f is constant. Suppose that $f(\perp) = y_0 \in Y$ and let $x \in X$. then $\perp \sqsubseteq x$, so $y_0 = f(\perp) \sqsubseteq f(x)$, so $f(x) = y_0$ for all $x \in X_\perp$ and f is constant.

End answer..

6. Show that the following two definitions of the ordering between continuous functions $f, g : D \to E$ (see Slide 35) are equivalent.

   (a) $f \sqsubseteq g := \forall\, d \in D(f(d) \sqsubseteq_E g(d))$.
   (b) $f \sqsubseteq' g := \forall\, d_1, d_2 \in D(d_1 \sqsubseteq_D d_2 \Rightarrow f(d_1) \sqsubseteq_E g(d_2))$.

Answer: ....
Obviously, if $f \sqsubseteq' g$, then $f \sqsubseteq g$: just take $d_1 = d_2 = d$ in the definition of $f \sqsubseteq' g$.

The other way around: suppose $f \sqsubseteq g$ and let $d_1, d_2 \in D$ with $d_1 \sqsubseteq_D d_2$. Then $f(d_1) \sqsubseteq g(d_1) \sqsubseteq g(d_2)$ by monotonicity of g, so done.

End answer . . . .