# Semantics and Domain theory

**Exercises 7**

1. Show that, if $Q \Downarrow_\tau V$, then $(\mathbf{fn}\,x : \tau.\,\mathbf{fn}\,y : \tau.y)PQ \Downarrow V$. (NB. $V$ denotes an arbitrary *value*.)
   **Answer:** ..............................................................
   Suppose we have a derivation of $Q \Downarrow_\tau V$. Then we have the following derivation of $(\mathbf{fn}\,x : \tau.\,\mathbf{fn}\,y : \tau.y)PQ \Downarrow V$, where we write $F$ for $\mathbf{fn}\,x : \tau.\,\mathbf{fn}\,y : \tau.y$.

$$\dfrac{\dfrac{F \Downarrow \mathbf{fn}\,x : \tau.\,\mathbf{fn}\,y : \tau.y \quad (\mathbf{fn}\,y : \tau.y)[P/x] \Downarrow \mathbf{fn}\,y : \tau.y}{FP \Downarrow \mathbf{fn}\,y : \tau.y}\Downarrow_{\text{app}} \qquad y[Q/y] \Downarrow V}{FPQ \Downarrow V}\Downarrow_{\text{app}}$$

   **End answer** ..............................................................

2. To prove that PCF evaluation is deterministic, we prove (in Proposition 5.4.1) that the following set is closed under the rules of Fig.3

$$\{(M, \tau, V) | M \Downarrow_\tau V \wedge \forall V'(M \Downarrow_\tau V' \Rightarrow V = V')\}$$

   Show this for the cases of the rules $(\Downarrow_{\text{if1}})$ and $(\Downarrow_{\text{cbn}})$.
   (An alternative way of looking at this is to prove the following:

$$M \Downarrow_\tau V \Rightarrow \forall V'(M \Downarrow_\tau V' \Rightarrow V = V')$$

   by induction on the derivation of $M \Downarrow_\tau V$. Do only the cases when the last applied rule is $(\Downarrow_{\text{if1}})$ or $(\Downarrow_{\text{cbn}})$.)

   **Answer:** ..............................................................
   The case for $\Downarrow_{\text{if1}}$:

$$\dfrac{M \Downarrow \mathbf{true} \quad N \Downarrow V}{\mathbf{if}\ M\ \mathbf{then}\ N\ \mathbf{else}\ P \Downarrow V}\Downarrow_{\text{if1}}$$

   We have the induction hypothesis for $M \Downarrow \mathbf{true}$ and for $N \Downarrow V$.
   If we have a derivation of $\mathbf{if}\ M\ \mathbf{then}\ N\ \mathbf{else}\ P \Downarrow V'$, then the last applied rules must be $\Downarrow_{\text{if1}}$ or $\Downarrow_{\text{if2}}$. As $M \Downarrow \mathbf{true}$, it follows from (IH) that $M \not\Downarrow \mathbf{false}$. So the last rule has been $\Downarrow_{\text{if1}}$ and we have

$$\dfrac{M \Downarrow \mathbf{true} \quad N \Downarrow V'}{\mathbf{if}\ M\ \mathbf{then}\ N\ \mathbf{else}\ P \Downarrow V'}\Downarrow_{\text{if1}}$$

   Now, (IH) for $N \Downarrow V$ says that $V = V'$ and we are done.
   The case for $\Downarrow_{\text{cbn}}$:

$$\dfrac{M \Downarrow \mathbf{fn}\,x : \tau.\,N \quad N[Q/x] \Downarrow V}{M\,Q \Downarrow V}\Downarrow_{\text{cbn}}$$

We have the induction hypothesis for $M \Downarrow \mathbf{fn}\, x : \tau.\, N$ and for $N[Q/x] \Downarrow V$. If we have a derivation of $M\, Q \Downarrow V'$, then the last applied rules must be $\Downarrow_{\text{cbn}}$. So we have

$$\frac{M \Downarrow \mathbf{fn}\, y : \tau'.\, N' \quad N'[Q/y] \Downarrow V'}{M\, Q \Downarrow V'} \Downarrow_{\text{cbn}}$$

Now, (IH) for $M \Downarrow \mathbf{fn}\, x : \tau.\, N$ says that $\mathbf{fn}\, x : \tau.\, N = \mathbf{fn}\, y : \tau'.\, N'$ and so $N[Q/x] = N'[Q/y]$. We are done by the (IH) for $N[Q/x] \Downarrow V$.
**End answer** ........................................................

3. Prove that the following terms $M$ and $N$ are not contextually equivalent.

   (a) $M = \mathbf{if}\ x\ \mathbf{then}\ 0\ \mathbf{else}\ 1$ and $N = \mathbf{if}\ y\ \mathbf{then}\ 0\ \mathbf{else}\ 1$.

   **Answer:** ........................................................
   Take $C[-] = (\mathbf{fn}\ x : \mathbf{bool}.\, \mathbf{fn}\, y : \mathbf{bool}.\, -)\, \mathbf{false}\, \mathbf{true}$. Then $C[M] \Downarrow 1$ and $C[N] \Downarrow 0$. The derivation of the first, writing
   $F$ for $\mathbf{fn}\, x : \mathbf{bool}.\, \mathbf{fn}\, y : \mathbf{bool}.\, \mathbf{if}\ x\ \mathbf{then}\ 0\ \mathbf{else}\ 1$ and
   $G$ for $\mathbf{fn}\, y : \mathbf{bool}.\, \mathbf{if\ false\ then}\ 0\ \mathbf{else}\ 1$ and

$$\frac{\dfrac{F \Downarrow F \quad \mathbf{fn}\, y : \mathbf{bool}.\, \mathbf{if}\ x\ \mathbf{then}\ 0\ \mathbf{else}\ 1[\mathbf{false}/x] \Downarrow G}{F\, \mathbf{false} \Downarrow G} \qquad \dfrac{\mathbf{false} \Downarrow \mathbf{false} \quad 1 \Downarrow 1}{\mathbf{if\ false\ then}\ 0\ \mathbf{else}\ 1 \Downarrow 1}}{F\, \mathbf{false}\, \mathbf{true} \Downarrow 1}$$

   Also, one could take $C[-] = (\mathbf{fn}\, x : \mathbf{bool}.\, \mathbf{fn}\, y : \mathbf{bool}.\, -)\ \Omega_{\mathbf{bool}}\ \mathbf{true}$. Then $C[M] \not\Downarrow$ and $C[N] \Downarrow 0$.
   **End answer** ........................................................

   (b) $M = \mathbf{fn}\, x : \mathbf{nat}.\mathbf{succ}(\mathbf{pred}\, x)$ and $N = \mathbf{fn}\, x : \mathbf{nat}.x$.

   **Answer:** ........................................................
   Take $C[-] = -\,0$. Then $C[M] \not\Downarrow$ and $C[N] \Downarrow 0$. The derivation of the second is easy, and we don't give it. For the first, suppose it has a derivation. Then show that this derivation should contain a subderivation of $\mathbf{pred}\, 0 \Downarrow V$ for some value $V$, but that derivation doesn't exist.
   **End answer** ........................................................

4. (a) Give a type $\tau$, a term $M$, values $V$, $V'$ and a context $C[-]$ such that $M \Downarrow_\tau V$ but $C[M] \Downarrow_\tau V' \neq C[V]$.
   **Answer:** ........................................................
   (So $M \Downarrow_\tau V \not\Rightarrow C[M] \Downarrow_\tau C[V]$.)
   A simple example is $M = \mathbf{pred}(\mathbf{succ}\, 0)$, $C[-] = \mathbf{fn}\, x : \mathbf{nat}.\, -$, because we have $M \Downarrow 0$ and $C[M] = \mathbf{fn}\, x : \mathbf{nat}.\, \mathbf{pred}(\mathbf{succ}\, 0) \Downarrow \mathbf{fn}\, x : \mathbf{nat}.\, \mathbf{pred}(\mathbf{succ}\, 0) \neq \mathbf{fn}\, x : \mathbf{nat}.\, 0 = C[0]$.
   NB. We don't give the derivations, but in an exam you should give them.
   **End answer** ........................................................

   (b) Give a type $\tau$, a term $M$, a value $V$ and a context $C[-]$ such that $M \Downarrow_\tau V$ but $C[M] \not\Downarrow_\tau$ ($C[M]$ has no value.)
   **Answer:** ........................................................

(So $M \Downarrow_\tau V \not\Rightarrow \exists V'(C[M] \Downarrow_\tau V')$.)
A very simple example is $M = 0$, $C[-] = \mathbf{pred}(-)$, because we have $M \Downarrow 0$ and $C[M] = \mathbf{pred}(0) \not\Downarrow$. A different example is $M = 0$, $C[-] = \mathbf{if\ zero}(-)\ \mathbf{then}\ \Omega_{\mathbf{nat}}\ \mathbf{else}\ -$, because we have $M \Downarrow 0$ and $C[M] = \mathbf{if\ zero}(0)\ \mathbf{then}\ \Omega_{\mathbf{nat}}\ \mathbf{else}\ 0 \not\Downarrow$.
NB. We don't give the derivations, but in an exam you should give them.
**End answer** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

(c) Give a type $\tau$, a term $M$, a value $V$ and a context $C[-]$ such that $M \not\Downarrow_\tau$ but $C[M] \Downarrow_\tau V$
**Answer:** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
(So $M \not\Downarrow_\tau \not\Rightarrow C[M] \not\Downarrow_\tau$).)
A very simple example is $M = \Omega_{\mathbf{nat}}$, $C[-] = \mathbf{if\ false\ then}\ -\ \mathbf{else}\ 0$, because we have $M \not\Downarrow$ and $C[M] = \mathbf{if\ false\ then}\ \Omega_{\mathbf{nat}}\ \mathbf{else}\ 0 \Downarrow 0$.
NB. We don't give the derivations, but in an exam you should give them.
**End answer** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

5. Given the definition of plus (Exercise 5.6.3.)

$$\begin{aligned}\text{plus} \quad = \quad & \mathbf{fix}(\mathbf{fn}\, p : \mathbf{nat} \to \mathbf{nat} \to \mathbf{nat}.\, \mathbf{fn}\, x : \mathbf{nat}.\, \mathbf{fn}\, y : \mathbf{nat}. \\ & \ \mathbf{if\ zero}(y)\ \mathbf{then}\ x\ \mathbf{else\ succ}(p\ x\ \mathbf{pred}(y)))\end{aligned}$$

Prove (by induction) that

$$\forall m, n(\text{plus}\ \mathbf{succ}^m(0)\ \mathbf{succ}^n(0) \ \Downarrow_{\mathbf{nat}}\ \mathbf{succ}^{m+n}(0))$$

NB. First identify the proper statement that you need and that you can prove relatively easily by induction.

**Answer:** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
We introduce some abbreviations

$$\begin{aligned}\underline{n} \quad &:= \quad \mathbf{succ}^n(0), \text{ for } n \in \mathbb{N} \\ A(p,x,y) \quad &:= \quad \mathbf{if\ zero}(y)\ \mathbf{then}\ x\ \mathbf{else\ succ}(p\ x\ \mathbf{pred}(y)) \\ B(p) \quad &:= \quad \mathbf{fn}\, x : \mathbf{nat}.\, \mathbf{fn}\, y : \mathbf{nat}. A(p,x,y) \\ H \quad &:= \quad \mathbf{fn}\, p : \mathbf{nat} \to \mathbf{nat} \to \mathbf{nat}. B(p) \\ T(y) \quad &:= \quad A(\text{plus}, \underline{m}, y)\end{aligned}$$

We prove, by induction on $n$, for all $Q$,

$$Q \Downarrow \underline{n} \Longrightarrow T(Q) \Downarrow \underline{m+n}.$$

So, $T(Q)$ is $\mathbf{if\ zero}(Q)\ \mathbf{then}\ \underline{m}\ \mathbf{else\ succ}(\text{plus}\ \underline{m}\ \mathbf{pred}(Q))$ and plus is defined as $\mathbf{fix}\, H$.

Case $n = 0$: immediate by one application of $(\Downarrow_{\text{if1}})$.

Case $n+1$: The IH is $\forall Q.(Q \Downarrow \underline{n} \Rightarrow T(Q) \Downarrow \underline{m+n}$. Now, suppose that $Q \Downarrow \underline{n+1}$ we need to prove $T(Q) \Downarrow \underline{m+n+1}$.

We use the fact that, if $Q \Downarrow \underline{n+1}$, then $\mathbf{pred}(Q) \Downarrow \underline{n}$.

The derivation is as follows: (You have to fill in the dots yourself and the double line requires two steps.)

$$
\dfrac{\dots \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dots}{H\ \text{plus} \Downarrow B(\text{plus})}}{\text{plus} \Downarrow B(\text{plus})} \qquad \dfrac{\text{IH}}{T(\mathbf{pred}(Q)) \Downarrow \underline{m+n}}}{\text{plus}\ \underline{m}\ \mathbf{pred}(Q) \Downarrow \underline{m+n}}}{\mathbf{succ}(\text{plus}\ \underline{m}\ \mathbf{pred}(Q)) \Downarrow \underline{m+n+1}}}{T(Q) \Downarrow \underline{m+n+1}}
$$

Now, the result, plus $\underline{m}\ \underline{n} \Downarrow \underline{m+n}$), follows immediately:

$$
\dfrac{\dfrac{\dfrac{\dots}{H\ \text{plus} \Downarrow B(\text{plus})}}{\text{plus} \Downarrow B(\text{plus})} \qquad \dfrac{\text{property about } T}{A(\text{plus}, \underline{m}, \underline{n}) \Downarrow \underline{m+n}}}{\text{plus}\ \underline{m}\ \underline{n} \Downarrow \underline{m+n}}
$$

**End answer** ...........................................................