Semantics and Domain theory

Exercises 8

1. (Exercise 6.5.2. of Pitts' notes) Define $\Omega_\tau = \mathbf{fix}(\mathbf{fn}\, x : \tau.x)$

   (a) Show that $[\![\Omega_\tau]\!]$ is the least element of the domain $[\![\tau]\!]$.
   Answer: ....
   $[\![\Omega_\tau]\!] = [\![\mathbf{fix}(\mathbf{fn}\, x : \tau.x)]\!] = \text{fix}(\boldsymbol{\lambda} d \in [\![\tau]\!].d) = \sqcup_{i\in\mathbb{N}}(\bot_{[\![\tau]\!]}) = \bot_{[\![\tau]\!]}$.
   End answer..

   (b) Deduce that $[\![\mathbf{fn}\, x : \tau.\Omega_\tau]\!] = [\![\Omega_{\tau\to\tau}]\!]$.
   Answer: ....
   $[\![\mathbf{fn}\, x : \tau.\Omega_\tau]\!] = \boldsymbol{\lambda} y \in [\![\tau]\!].[\![\Omega_\tau]\!] = \boldsymbol{\lambda} y \in [\![\tau]\!].\bot_{[\![\tau]\!]} = \bot_{[\![\tau\to\tau]\!]} = [\![\Omega_{\tau\to\tau}]\!]$.
   End answer..

2. (a) Compute the denotational semantics of
   $M = \mathbf{fn}\, x : \mathbf{bool}.\, \mathbf{fn}\, y : \mathbf{nat}.\mathbf{if}\; x\; \mathbf{then}\; y\; \mathbf{else}\; y$.
   Answer: ....

   $[\![M]\!] = \boldsymbol{\lambda} b \in \mathbb{B}_\bot.\boldsymbol{\lambda} c \in \mathbb{N}_\bot.\text{if}(b, c, c) =$

   $$\begin{cases} \bot & \text{if } b = \bot \\ c & \text{if } b \neq \bot \end{cases}$$

   End answer..

   (b) Define a term P such that $[\![M]\!] \sqsubseteq [\![P]\!]$ but $[\![M]\!] \neq [\![P]\!]$.
   Answer: ....
   $P := \mathbf{fn}\, x : \mathbf{bool}.\, \mathbf{fn}\, y : \mathbf{nat}.y$. Then $[\![\mathbf{fn}\, x : \mathbf{bool}.\, \mathbf{fn}\, y : \mathbf{nat}.y]\!] = \boldsymbol{\lambda} b \in \mathbb{B}_\bot.\boldsymbol{\lambda} c \in \mathbb{N}_\bot.c$ and $[\![M]\!] \sqsubseteq [\![P]\!]$, $[\![M]\!] \neq [\![P]\!]$.
   End answer..

   (c) Define a term N such that $[\![M]\!] = [\![N]\!]$ but $N \not\Downarrow M$.

   Answer: ....
   $N = \mathbf{fn}\, x : \mathbf{bool}.\, \mathbf{fn}\, y : \mathbf{nat}.\mathbf{if}\; x\; \mathbf{then}\; y\; \mathbf{else}\; (\mathbf{if}\; x\; \mathbf{then}\; y\; \mathbf{else}\; y)$. Then M and N are both values, so $N \not\Downarrow M$ and $M \not\Downarrow N$. On the other hand, $[\![N]\!] = [\![M]\!]$.
   End answer..

3. Define terms $M, N : \mathbf{nat} \to \mathbf{nat}$ with $[\![M]\!] \sqsubseteq [\![N]\!]$ and $[\![M]\!] \neq [\![N]\!]$.
   Answer: ....
   $N = \mathbf{fn}\, y : \mathbf{nat}.y$ and $M = \mathbf{fn}\, y : \mathbf{nat}.\mathbf{succ}(\mathbf{pred}(y))$. Then

   $$[\![M]\!] = \boldsymbol{\lambda} d \in \mathbb{N}_\bot \begin{cases} \bot & \text{if } d = \bot, 0 \\ d & \text{if } d \neq \bot, 0 \end{cases}$$

   So $[\![M]\!] \sqsubseteq [\![N]\!]$ and $[\![M]\!] \neq [\![N]\!]$.
   End answer..

4. Verify that $[\![(\mathbf{fn}\, x : \sigma.M)N]\!](\rho) = [\![M[N/x]]\!](\rho)$ for M, N with $\Gamma \vdash N : \sigma$ and $\Gamma, x : \sigma \vdash M : \tau$ and $\rho \in [\![\Gamma]\!]$. (Use the result on Slide 62, the Substitution Lemma.)

   Answer: ....

$$\begin{aligned}
[\![(\mathbf{fn}\, x:\sigma.M)N]\!](\rho) &= [\![\mathbf{fn}\, x:\sigma.M]\!](\rho))[\![N]\!](\rho) \\
&= (\boldsymbol{\lambda} d \in [\![\sigma]\!].[\![M]\!](\rho[x \mapsto d]))[\![N]\!](\rho) \\
&= [\![M]\!](\rho[x \mapsto [\![N]\!](\rho)]) \\
&= [\![M[N/x]]\!](\rho)
\end{aligned}$$

End answer. . . . .

5. Prove Soundness of the operational semantics (Theorem 6.4.1) for the inductive cases $\Downarrow_{\text{pred}}$ and $\Downarrow_{\text{if1}}$.
Remember that Theorem 6.4.1 states that for all closed expressions M and V and type τ, if $M \Downarrow_\tau V$, then $[\![M]\!] = [\![V]\!]$. It is proved by induction on the derivation of $M \Downarrow_\tau V$.

Answer: . . . .

The case for $\Downarrow_{\text{pred}}$:

$$\frac{M \Downarrow \mathbf{succ}(V)}{\mathbf{pred}(M) \Downarrow V} \Downarrow_{\text{pred}}$$

$\mathbf{succ}(V) : \mathbf{nat}$, so $V = \mathbf{succ}^p(0)$ for some $p \in \mathbb{N}$. We have the induction hypothesis for $M \Downarrow \mathbf{succ}(V)$, so for all ρ, $[\![M]\!](\rho) = [\![\mathbf{succ}(V)]\!](\rho) = [\![\mathbf{succ}(\mathbf{succ}^p(0))]\!](\rho) = p+1$. Then $[\![\mathbf{pred}(M)]\!](\rho) = p = [\![V]\!](\rho)$ (for all ρ).

The case for $\Downarrow_{\text{if1}}$:

$$\frac{M \Downarrow \mathbf{true} \quad N \Downarrow V}{\mathbf{if}\ M\ \mathbf{then}\ N\ \mathbf{else}\ P \Downarrow V} \Downarrow_{\text{if1}}$$

We have the induction hypothesis for $M \Downarrow \mathbf{true}$ and for $N \Downarrow V$, so for all ρ, $[\![M]\!](\rho) = \text{tt}$ and $[\![N]\!](\rho) = [\![V]\!](\rho)$. Then, by definition of $[\![-]\!]$ and IH: $[\![\mathbf{if}\ M\ \mathbf{then}\ N\ \mathbf{else}\ P]\!](\rho) = [\![N]\!](\rho) = [\![V]\!](\rho)$ (for all ρ).

End answer. . . . .

6. Let

$P := \mathbf{fix}(\mathbf{fn}\, p : \mathbf{nat} \to \mathbf{bool}.\, \mathbf{fn} x : \mathbf{nat}.\, \mathbf{if}\ (\mathbf{zero}\, x)\ \mathbf{then\ true\ else}\ p(\mathbf{pred}\,(\mathbf{pred}\, x)))$.

Compute $[\![P]\!]$.

Answer: . . . .
Define $H := \mathbf{fn}\, p : \mathbf{nat} \to \mathbf{bool}.\, \mathbf{fn} x : \mathbf{nat}.\, \mathbf{if}\ (\mathbf{zero}\, x)\ \mathbf{then\ true\ else}\ p(\mathbf{pred}\,(\mathbf{pred}\, x))$.

We define $h := [\![H]\!] = \boldsymbol{\lambda} f \in \mathbb{N}_\perp \to \mathbb{B}_\perp.\boldsymbol{\lambda} d \in \mathbb{N}_\perp.\text{if}(d = 0, \text{tt}, f(d \ominus 2))$, where $d \ominus 2$ is defined as $\perp$ if $d = \perp, 0, 1$ and $d - 2$ if $d \geq 2$.

$$\begin{aligned}
[\![P]\!] &= \mathbf{fix}\, [\![H]\!] \\
&= \sqcup_{\imath \in \mathbb{N}} h^i(\perp_{\mathbb{N}_\perp \to \mathbb{B}_\perp}) \\
&\overset{*}{=} \boldsymbol{\lambda} d \in \mathbb{N}_\perp. \begin{cases} \text{tt} & \text{if } d \in \mathbb{N} \text{ is even} \\ \perp & \text{if } d = \perp \text{ or } d \in \mathbb{N} \text{ is odd.} \end{cases}
\end{aligned}$$

The proof of $\stackrel{*}{=}$ is by showing that for $g_i := h^i(\bot_{\mathbb{N}_\bot \to \mathbb{B}_\bot})$ we have $g_i(2d) = \mathrm{tt}$ if $d \in \mathbb{N}$ with $d < i$ and $g_i(d) = \bot$ otherwise. (Proof by induction on i.)

End answer ....

7. We give PCF-terms that define the "or" function, taking partiality into account. A PCF-term M *defines* the domain-theoretic function f if $[\![M]\!] = f$.

   (a) Give a PCF-term that defines the function $f : \mathbb{B}_\bot \to \mathbb{B}_\bot \to \mathbb{B}_\bot$ given by

   $$f(x)(y) := \begin{cases} \bot & \text{if } x = \bot \text{ or } (x = \mathrm{ff} \text{ and } y = \bot) \\ \mathrm{tt} & \text{if } x = \mathrm{tt} \text{ or } (x = \mathrm{ff} \text{ and } y = \mathrm{tt}) \\ \mathrm{ff} & \text{if } x = \mathrm{ff} \text{ and } y = \mathrm{ff} \end{cases}$$

   Answer: ....
   Take $M :=$ **fn** x : **bool. fn** y : **bool. if** x **then true else** y. Compute for yourself $[\![M]\!]$ and observe that $[\![M]\!] = f$.
   End answer ....

   (b) Give a PCF-terms that defines the function $g : \mathbb{B}_\bot \to \mathbb{B}_\bot \to \mathbb{B}_\bot$ given by

   $$g(x)(y) := \begin{cases} \bot & \text{if } x = \bot \text{ or } y = \bot \\ \mathrm{tt} & \text{if } x, y \neq \bot \text{ and } (x = \mathrm{tt} \text{ or } y = \mathrm{tt}) \\ \mathrm{ff} & \text{if } x = \mathrm{ff} \text{ and } y = \mathrm{ff} \end{cases}$$

   Answer: ....
   Take $N :=$
   fn x : **bool. fn** y : **bool. if** x **then** (**if** y **then true else true**) **else** y.
   Compute for yourself $[\![N]\!]$ and observe that $[\![N]\!] = g$.
   End answer ....