

Semantics and Domain theory

Exercises 9

1. Prove the following properties (by induction on τ). Here, M, M_1, M_2 range over closed terms, d_1, d_2 are domain elements.

- (a) If $d_2 \sqsubseteq d_1$ and $d_1 \triangleleft_\tau M_1$, then $d_2 \triangleleft_\tau M_1$.

Answer:

The proof is by induction on τ .

Case $\tau = \mathbf{nat}$: From $d_1 \triangleleft_{\mathbf{nat}} M_1$ we derive (1) $d_1 = \perp$ or (2) $d_1 \in \mathbb{N}$ and $M_1 \Downarrow \mathbf{succ}^{d_1}(0)$. Now, if $d_2 \sqsubseteq d_1$, then, in case (1): $d_2 = \perp$, so $d_2 \triangleleft_\tau M_1$ and in case (2) $d_2 = \perp$ or $d_2 = d_1 \in \mathbb{N}$ and then also $d_2 \triangleleft_{\mathbf{nat}} M_1$.

Case $\tau = \mathbf{bool}$ is similar to the **nat**-case.

Case $\tau = \sigma_1 \rightarrow \sigma_2$. From $d_1 \triangleleft_\tau M_1$ we derive: $\forall e, N (e \triangleleft_{\sigma_1} N \Rightarrow d_1(e) \triangleleft_{\sigma_2} M_1 N)$. Now let e, N be such that $e \triangleleft_{\sigma_1} N$. (To prove: $d_2(e) \triangleleft_{\sigma_2} M_1 N$.) From $d_2 \sqsubseteq d_1$ we derive $d_2(e) \sqsubseteq d_1(e)$ and by induction hypothesis we find that $d_2(e) \triangleleft_{\sigma_2} M_1 N$ and we are done.

End answer

- (b) If $d_1 \triangleleft_\tau M_1$ and $\forall V (M_1 \Downarrow_\tau V \Rightarrow M_2 \Downarrow_\tau V)$, then

$$d_1 \triangleleft_\tau M_2$$

Answer:

The proof is by induction on τ .

Case $\tau = \mathbf{nat}$: From $d_1 \triangleleft_{\mathbf{nat}} M_1$ we derive (1) $d_1 = \perp$ or (2) $d_1 \in \mathbb{N}$ and $M_1 \Downarrow \mathbf{succ}^{d_1}(0)$. In case (1): $d_1 \triangleleft_{\mathbf{nat}} M_2$ is immediate. In case (2) we derive $M_2 \Downarrow \mathbf{succ}^{d_1}(0)$ from the hypothesis, so $d_1 \triangleleft_\tau M_2$.

Case $\tau = \mathbf{bool}$ is similar to the **nat**-case.

Case $\tau = \sigma_1 \rightarrow \sigma_2$. Suppose $d_1 \triangleleft_{\sigma_1 \rightarrow \sigma_2} M_1$ (*) and $\forall V (M_1 \Downarrow_{\sigma_1 \rightarrow \sigma_2} V \Rightarrow M_2 \Downarrow_{\sigma_1 \rightarrow \sigma_2} V)$ (**). Suppose $e \triangleleft_{\sigma_1} N$ (To prove: $d_1(e) \triangleleft_{\sigma_2} M_2 N$.) From (*) we derive $d_1(e) \triangleleft_{\sigma_2} M_1 N$. By IH:

$$\forall V (M_1 N \Downarrow_{\sigma_2} V \Rightarrow M_2 N \Downarrow_{\sigma_2} V) \Rightarrow d_1(e) \triangleleft_{\sigma_2} M_2 N,$$

So we are done if we prove $\forall V (M_1 N \Downarrow_{\sigma_2} V \Rightarrow M_2 N \Downarrow_{\sigma_2} V)$.

Suppose $M_1 N \Downarrow_{\sigma_2} V$. This can only be the case if $M_1 \Downarrow_{\sigma_1 \rightarrow \sigma_2} \mathbf{fn} x : \sigma_1.P$ for some P . By (**) we derive that $M_2 \Downarrow_{\sigma_1 \rightarrow \sigma_2} \mathbf{fn} x : \sigma_1.P$ and so $M_2 N \Downarrow_{\sigma_2} V$.

End answer

These properties constitute Lemma 7.2.1 (iii).

2. Remember that \triangleleft_τ denotes the approximation relation (Slide 64). Show that, if $d \triangleleft_{\mathbf{nat}} M$, $e \triangleleft_{\mathbf{nat}} N$ and $b \triangleleft_{\mathbf{bool}} P$, then

$$\mathbf{if}(b, d, e) \triangleleft_{\mathbf{nat}} \mathbf{if} P \mathbf{then} M \mathbf{else} N$$

(This is basically the "if" inductive case in the proof of the Fundamental Property, Slide 65)

Answer:

We need to show that, in case $\mathbf{if}(b, d, e) = n \in \mathbb{N}$, then $\mathbf{if} P \mathbf{then} M \mathbf{else} N \Downarrow \mathbf{succ}^n(0)$. So suppose that $\mathbf{if}(b, d, e) = n \in \mathbb{N}$. Then either (1) $b = \mathbf{tt}$ and

$d = n$ or (2) $b = \text{ff}$ and $e = n$. In case (1), $P \Downarrow \text{true}$ (by $b \triangleleft_{\text{bool}} P$) and $M \Downarrow \text{succ}^n(0)$ (by $d \triangleleft_{\text{nat}} M$), so $\text{if } P \text{ then } M \text{ else } N \Downarrow \text{succ}^n(0)$. Case (2) is similar.

End answer

3. Prove that $\text{fn } x : \text{nat.succ(pred } x) \leq_{\text{ctx}} \text{fn } x : \text{nat.}x$ in the following two ways:

- (a) By using the Proposition on Slide 68.

Answer:

We need to show that $\llbracket \text{fn } x : \text{nat.succ(pred } x) \rrbracket \triangleleft_{\text{nat} \rightarrow \text{nat}} \llbracket \text{fn } x : \text{nat.}x \rrbracket$.

We first compute $f := \llbracket \text{fn } x : \text{nat.succ(pred } x) \rrbracket \in \mathbb{N}_{\perp} \rightarrow \mathbb{N}_{\perp}$:

$$f(d) := \begin{cases} \perp & \text{if } d = \perp, 0 \\ d & \text{otherwise} \end{cases}$$

Now we have to show that $\forall d, N (d \triangleleft_{\text{nat}} N \Rightarrow f(d) \triangleleft_{\text{nat}} (\text{fn } x : \text{nat.}x)N)$.

Suppose $d \triangleleft_{\text{nat}} N$. If $f(d) = \perp$, we are done because $\perp \triangleleft_{\text{nat}} P$ for all $P : \text{nat}$.

If $f(d) = d$, then $d \in \{1, 2, \dots\}$, so $d \triangleleft_{\text{nat}} N$ implies $N \Downarrow \text{succ}^d(0)$.

But then $(\text{fn } x : \text{nat.}x)N \Downarrow \text{succ}^d(0)$ and we are done.

End answer

- (b) By using the Extensionality properties on Slide 69.

Answer:

We need to show that

$$\forall M, N : \text{nat} (M \leq_{\text{ctx}} N \Rightarrow (\text{fn } x : \text{nat.succ(pred } x))M \leq_{\text{ctx}} (\text{fn } x : \text{nat.}x)N).$$

Suppose $M \leq_{\text{ctx}} N$, so $\forall V (M \Downarrow V \Rightarrow N \Downarrow V)$.

Now, suppose that for V , we have $(\text{fn } x : \text{nat.succ(pred } x))M \Downarrow V$.

(To prove: $(\text{fn } x : \text{nat.}x)N \Downarrow V$.) Then, by analyzing the derivation

of $(\text{fn } x : \text{nat.succ(pred } x))M \Downarrow V$, we observe that $V = \text{succ}^{n+1}(0)$

for some $n \in \mathbb{N}$ and $M \Downarrow \text{succ}^{n+1}(0)$. Then also $N \Downarrow \text{succ}^{n+1}(0)$ and

$(\text{fn } x : \text{nat.}x)N \Downarrow V$ and we are done.

End answer

4. Consider the terms $M_1 := \text{fix}(\text{fn } f : \text{nat} \rightarrow \text{nat.}f)$ and

$M_2 := \text{fn } x : \text{nat.fix}(\text{fn } y : \text{nat.}y)$ of type $\text{nat} \rightarrow \text{nat}$. Use the Extensionality property of \leq_{ctx} at function types (Slide 69) to show that $M_1 \cong_{\text{ctx}} M_2$.

Answer:

We need to prove that

$$\forall P, Q : \text{nat} (P \leq_{\text{ctx}} Q \Rightarrow M_1 P \leq_{\text{ctx}} M_2 Q),$$

and the other way around:

$$\forall P, Q : \text{nat} (P \leq_{\text{ctx}} Q \Rightarrow M_2 P \leq_{\text{ctx}} M_1 Q).$$

We only do the first, as the second is similar. So let $P, Q : \text{nat}$ with $P \leq_{\text{ctx}} Q$.

Now, suppose $M_1 P \Downarrow V$. (To prove: $M_2 Q \Downarrow V$.) By analyzing the possible

derivation of $M_1 P \Downarrow V$, we observe that it doesn't exist. (Because we would

need $M_1 \Downarrow \text{fn } x : \text{nat.}R$ for some R , but M_1 doesn't evaluate; M_1 is the

well-known term Ω .) So we have a contradiction from the assumption, so the

conclusion holds vacuously and we are done. (For the other way around, we

analyze the possible derivation of $M_2 P \Downarrow V$, which doesn't exist either.)

End answer

5. Prove that for all $M_1, M_2 \in \text{PCF}_\tau$,

$M_1 \leq_{\text{ctx}} M_2 : \tau$ if and only if

$\forall M \in \text{PCF}_{\tau \rightarrow \text{bool}}(M M_1 \Downarrow_{\text{bool}} \text{true} \Rightarrow M M_2 \Downarrow_{\text{bool}} \text{true})$.

(Remember that PCF_τ are the *closed* PCF-terms of type τ .)

Answer:
 The “only if” is clear: If $M_1 \leq_{\text{ctx}} M_2 : \tau$, take $C[-] = M -$ (with $M \in \text{PCF}_{\tau \rightarrow \text{bool}}$) and the conclusion follows.

For the “if” direction, suppose

$$\forall M \in \text{PCF}_{\tau \rightarrow \text{bool}}(M M_1 \Downarrow_{\text{bool}} \text{true} \Rightarrow M M_2 \Downarrow_{\text{bool}} \text{true})$$

and let $C[-]$ be a term-with-hole with $C[M_1], C[M_2] : \mathbf{nat}$. (The case for **bool** is similar.) We need to prove that, for all V , if $C[M_1] \Downarrow V$, then $C[M_2] \Downarrow V$, so let V be a value with $C[M_1] \Downarrow V$, say $V = \text{succ}^n(0)$. Now take $M : \mathbf{nat} \rightarrow \mathbf{bool}$ as follows:

$$M := \mathbf{fn} x : \tau. \text{zero}(\text{pred}^n(C[x])).$$

We now have

$$\begin{aligned} C[M_1] \Downarrow \text{succ}^n(0) &\Rightarrow M M_1 \Downarrow \text{true} \\ &\Rightarrow M M_2 \Downarrow \text{true} \\ &\Rightarrow C[M_2] \Downarrow \text{succ}^n(0) \end{aligned}$$

and we are done.

End answer

6. (Exercise 7.4.1.) For any PCF type τ and closed terms M_1, M_2 of type τ , show that

$$(\forall V : \tau, (M_1 \Downarrow_\tau V \Leftrightarrow M_2 \Downarrow_\tau V)) \Rightarrow M_1 \cong_{\text{ctx}} M_2 : \tau. \quad (**)$$

[Hint: combine the Proposition on Slide 68 with Exercise 1 above (or Lemma 7.2.1(iii)).]

Answer:

We prove the following, which suffices

$$(\forall V : \tau, (M_1 \Downarrow_\tau V \Rightarrow M_2 \Downarrow_\tau V)) \Rightarrow M_1 \leq_{\text{ctx}} M_2 : \tau.$$

From the Proposition on Slide 68, we derive: $\llbracket M_1 \rrbracket \triangleleft_\tau M_2 \Rightarrow M_1 \leq_{\text{ctx}} M_2 : \tau$, so we are done if we derive $\llbracket M_1 \rrbracket \triangleleft_\tau M_2$ from $\forall V : \tau, (M_1 \Downarrow_\tau V \Rightarrow M_2 \Downarrow_\tau V)$. So suppose $\forall V : \tau, (M_1 \Downarrow_\tau V \Rightarrow M_2 \Downarrow_\tau V)$. By Exercise 1b, it suffices to prove $\llbracket M_1 \rrbracket \triangleleft_\tau M_1$, which holds by the Fundamental property for \triangleleft .

End answer

7. (Exercise 7.4.2.) For any PCF type τ and closed terms M_1, M_2 of type τ , we have

$$(\forall V : \tau, (M_1 \Downarrow_\tau V \Leftrightarrow M_2 \Downarrow_\tau V)) \Rightarrow M_1 \cong_{\text{ctx}} M_2 : \tau. \quad (**)$$

Use (**) to show that β -conversion is valid up to contextual equivalence in PCF, in the sense that for all closed terms $\mathbf{fn} x : \tau_1. P : \tau_1 \rightarrow \tau_2$ and $Q : \tau_1$,

$$(\mathbf{fn} x : \tau_1. P) Q \cong_{\text{ctx}} P[Q/x] : \tau_2.$$

Answer:
 By (**) it suffices to show that

$$\forall V : \tau((\mathbf{fn} x : \tau_1. P) Q \Downarrow_\tau V \Leftrightarrow P[Q/x] \Downarrow_\tau V).$$

Suppose $(\mathbf{fn} x : \tau_1. P) Q \Downarrow_\tau V$. The derivation of this judgment (write the derivation rule for \mathbf{fn}) contains a subderivation of $P[Q/x] \Downarrow_\tau V$, so we are done.

Suppose $P[Q/x] \Downarrow_\tau V$. Using a derivation of $P[Q/x] \Downarrow_\tau V$, we can immediately construct a derivation of $(\mathbf{fn} x : \tau_1. P) Q \Downarrow_\tau V$. (using the derivation rule for \mathbf{fn}), so we are done.

End answer

8. (Exercise 7.4.3.) Show that the converse of (**) is not valid at all types by considering the terms M_1 and M_2 of Exercise 4

Answer:

From Exercise 4, we derive that $M_1 \cong_{\text{ctx}} M_2$. On the other hand, M_2 is a value, so $M_2 \Downarrow M_2$, while $M_1 \not\Downarrow$.

End answer