Semantics and Domain theory

Exercises 10

Exercises from the dI-domains notes

1. Suppose that a monotonic function $p : (\mathbb{B}_\perp \times \mathbb{B}_\perp) \to \mathbb{B}_\perp$ satisfies

   - $p(\mathrm{tt}, \perp) = \mathrm{tt}$,
   - $p(\perp, \mathrm{tt}) = \mathrm{tt}$,
   - $p(\mathrm{ff}, \mathrm{ff}) = \mathrm{ff}$.

   Show that p coincides with the parallel-or function (on Slide 72 of the notes of Pitts) in the sense that $p(d_1, d_2) = \mathrm{por}(d_1)(d_2)$, for all $d_1, d_2 \in \mathbb{B}_\perp$.

2. Show that the evaluation relation for PCF+por (Slide 77, where rules for $\mathbf{por}(M_1, M_2) \Downarrow V$ have been added to PCF) is still deterministic: If $M \Downarrow V$, then for all V', if $M \Downarrow V'$, then $V = V'$. This is again proved by induction on the derivation of $M \Downarrow V$; do the case for the new **por**-rules.

3. (a) Describe the compact elements of $\mathbb{N}_\perp \to \mathbb{N}_\perp$ and prove that these are indeed the compact elements.

   (b) Show that $\mathbb{N}_\perp \to \mathbb{N}_\perp$ is an algebraic dcpo.

4. Let X be a set and let $\wp(X)$ be the power set of X ordered by inclusion $\subseteq$.

   (a) Describe the compact elements of $\wp(X)$ and prove that these are indeed the compact elements..

   (b) Show that $\wp(X)$ is an algebraic dcpo.

5. Suppose we are in a dcpo where each pair of elements has a glb. Show that $\forall x, y (x \sqsubseteq y \Leftrightarrow x = x \sqcap y)$.

   Answer: ....

   We always have $x \sqcap y \sqsubseteq x$. If $x \sqsubseteq y$, then $x \sqsubseteq y \wedge x \sqsubseteq x$, so $x \sqsubseteq x \sqcap y$, so we have $x = x \sqcap y$

   If $x = x \sqcap y$ then from $x \sqcap y \sqsubseteq y$ we conclude $x \sqsubseteq y$.

   End answer..

6. (a) Prove that $\mathbb{N}_\perp$ satisfies (axiom d):

   $$\forall x, y, z \in \mathbb{N}_\perp (y \uparrow z \Rightarrow x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)).$$

   Answer: ....

   Assume $y \uparrow z$. This means we have (1) $y = z$ or (2) $y = \perp, z \in \mathbb{N}$ or (3) $y \in \mathbb{N}, z = \perp$.

   In case (1) we have $x \sqcap (y \sqcup y) = (x \sqcap y) \sqcup (x \sqcap y)$, which holds.

   In case (2) we have $x \sqcap z = \perp \sqcup (x \sqcap z)$, which holds.

   Case (3) is similar to case (2)

   End answer..

   (b) Prove that $\mathbb{N}_\perp \to \mathbb{N}_\perp$ (the set of Scott continuous functions) satisfies (axiom d).

   Answer: ....

   First observe the following.

- $g \uparrow h$ means $\forall x \in \text{dom}(g) \cap \text{dom}(h)(g(x) = h(x))$.
- If $g \uparrow h$, $g \sqcup h$ is the function

$$(g \sqcup h)(x) = \begin{cases} g(x) & \text{if } g(x) \neq \bot \\ h(x) & \text{if } h(x) \neq \bot \\ \bot & \text{otherwise.} \end{cases}$$

- $f \sqcap g$ is the function

$$(f \sqcap g)(x) = \begin{cases} f(x) & \text{if } x \in \text{dom}(f) \cap \text{dom}(g) \wedge f(x) = g(x) \\ \bot & \text{otherwise.} \end{cases}$$

Now assume $g \uparrow h$. We have

$$\begin{aligned} (f \sqcap (g \sqcup h))(x) &= \begin{cases} f(x) & \text{if } x \in \text{dom}(f) \cap \text{dom}(g \sqcup h) \wedge f(x) = (g \sqcup h)(x) \\ \bot & \text{otherwise.} \end{cases} \\ &= \begin{cases} f(x) & \text{if } x \in \text{dom}(f) \cap (\text{dom}(g) \cup \text{dom}(h)) \wedge \\ & \quad f(x) = g(x) \text{ if } g(x) \neq \bot \wedge f(x) = h(x) \text{ if } h(x) \neq \bot \\ \bot & \text{otherwise.} \end{cases} \\ &= \begin{cases} f(x) & \text{if } (f(x) = g(x) \neq \bot) \vee f(x) = h(x) \neq \bot \\ \bot & \text{otherwise.} \end{cases} \end{aligned}$$

We also have

$$\begin{aligned} ((f \sqcap g) \sqcup (f \sqcap h))(x) &= \begin{cases} (f \sqcap g)(x) & \text{if } (f \sqcap g)(x) \neq \bot \\ (f \sqcap h)(x) & \text{if } (f \sqcap h)(x) \neq \bot \\ \bot & \text{otherwise.} \end{cases} \\ &= \begin{cases} f(x) & \text{if } x \in \text{dom}(f) \cap \text{dom}(g) \wedge f(x) = g(x) \\ f(x) & \text{if } x \in \text{dom}(f) \cap \text{dom}(h) \wedge f(x) = h(x) \\ \bot & \text{otherwise.} \end{cases} \\ &= \begin{cases} f(x) & \text{if } (f(x) = g(x) \neq \bot) \vee f(x) = h(x) \neq \bot \\ \bot & \text{otherwise.} \end{cases} \end{aligned}$$

So $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$.
End answer..

7. We are in a bounded complete p.o. and we consider the property $(*)$ (used in stability)

$$\forall x, y \in D(x \uparrow y \to f(x \sqcap y) = f(x) \sqcap f(y)) \qquad (*)$$

Show that, if f satisfies $(*)$, then it is monotone.

Answer: ....
Suppose $x \sqsubseteq y$. By Exercise 5 we have $x = x \sqcap y$. Also $x \uparrow y$, so we have: $f(x) = f(x \sqcap y) = f(x) \sqcap f(y)$. So $f(x) = f(x) \sqcap f(y)$ and again by Exercise 5 we conclude that $f(x) \sqsubseteq f(y)$.
End answer..

8. (a) Define all possible different "AND" functions as monotone functions (in $\mathbb{B}_\bot \times \mathbb{B}_\bot \to \mathbb{B}_\bot$).
Answer: ....
There are 4 different momotone "AND" functions. For each and function we have

   - $\mathsf{and}(\text{tt}, \text{tt}) = \text{tt}$,

- $\mathsf{and}(\mathrm{tt}, \mathrm{ff}) = \mathsf{and}(\mathrm{ff}, \mathrm{tt}) = \mathsf{and}(\mathrm{ff}, \mathrm{ff}) = \mathrm{ff}$,
- $\mathsf{and}(\bot, \bot) = \mathsf{and}(\mathrm{tt}, \bot) = \mathsf{and}(\bot, \mathrm{tt}) = \bot$.

Then we differentiate between them as follows.

i. $\mathsf{and}_1(\mathrm{ff}, \bot) = \mathrm{ff}$, $\mathsf{and}_1(\bot, \mathrm{ff}) = \bot$.
ii. $\mathsf{and}_2(\mathrm{ff}, \bot) = \bot$, $\mathsf{and}_2(\bot, \mathrm{ff}) = \mathrm{ff}$.
iii. $\mathsf{and}_3(\mathrm{ff}, \bot) = \bot$, $\mathsf{and}_3(\bot, \mathrm{ff}) = \bot$.
iv. $\mathsf{and}_4(\mathrm{ff}, \bot) = \mathrm{ff}$, $\mathsf{and}_4(\bot, \mathrm{ff}) = \mathrm{ff}$.

End answer..

(b) Show that 3 of your functions can be defined in PCF.
Answer: ....

i. and_1 is defined by **fn** x : **bool. fn** y : **bool. if** x **then** y **else false**. Compute for yourself that this indeed defines and_1.
ii. and_2 is defined by **fn** x : **bool. fn** y : **bool. if** y **then** x **else false**. Compute for yourself that this indeed defines and_2.
iii. and_3 is defined by
fn x : **bool. fn** y : **bool. if** x **then** y **else** (**if** y **then false else false**). Compute for yourself that this indeed defines and_3.

End answer..

(c) Now show that one of your functions cannot be defined in PCF

i. by semantic means (using dI-domains).
Answer: ....
The function and_4 is not stable, and hence it cannot be defined. That and_4 is not stable is seen as follows. We have $(\mathrm{ff}, \bot) \uparrow (\bot, \mathrm{ff})$, but

$$\mathrm{ff} = \mathsf{and}_4(\mathrm{ff}, \bot) \sqcap \mathsf{and}_4(\bot, \mathrm{ff}) \neq \mathsf{and}_4((\mathrm{ff}, \bot) \sqcap (\bot, \mathrm{ff})) = \bot.$$

End answer ....

ii. by using the non-definability of por

Answer: ....
The function and_4 cannot be defined, because if it could, say by the term M : **bool** $\to$ **bool** $\to$ **bool**, then we would also be able to define por (parallel or), by

$$P := \textbf{fn}\ x : \textbf{bool}.\,\textbf{fn}\ y : \textbf{bool}.\,\mathrm{neg}(M(\mathrm{neg}\ x)(\mathrm{neg}\ y)),$$

where neg := **fn** x : **bool. if** x **then false else true** defines negation. (Check this by computing the semantics of P.)
End answer ....

9. Prove that the identity function $I : \mathbb{N}_\bot \to \mathbb{N}_\bot$ is not very finite.
Answer: ....
Consider, for $i \in \mathbb{N}$,

$$f_i(x) := \begin{cases} x & \text{if } x \leq i \\ \bot & \text{if } x = \bot \text{ or } x > i \end{cases}$$

We have $f_i \sqsubseteq I$ and all the f_i are different, so $\{g \mid g \sqsubseteq I\}$ is not finite, so I is not very finite.
End answer..

10. Prove that, in a bounded complete dcpo D, every non-empty set X has a greatest lower bound, $\bigsqcap X$.

Answer: ....
Given $X \neq \emptyset$, there is a $x_0 \in X$, so $Y := \{d \in D \mid \forall x \in X(d \sqsubseteq x)\}$ is bounded (by x_0), so we can define

$$d_0 := \bigsqcup \{d \in D \mid \forall x \in X(d \sqsubseteq x)\}.$$

We claim that $d_0 = \bigsqcap X$. We need to prove the following two properties.

(glb1) To prove: $\forall x \in X(d_0 \sqsubseteq x)$.
Suppose $x \in X$. Then $\forall d \in Y(d \sqsubseteq x)$, so x is an upperboud of Y, so $d_0 \sqsubseteq x$, because d_0 is the least upperbound of Y.

(glb2) To prove: $\forall d(\forall x \in X(d \sqsubseteq x)) \Rightarrow d \sqsubseteq d_0$.
Suppose $\forall x \in X(d \sqsubseteq x)$. Then $d \in Y$, so $d \sqsubseteq d_0$, because d_0 is an upperbound of Y.

End answer ....