Natural deduction for propositional logic via truth tables

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Outline

Truth tables

Classically, the meaning of a propositional connective is fixed by its truth table. This immediately implies

- consistency,
- a decision procedure,
- completeness (w.r.t. Boolean algebra's).

Intuitionistically, the meaning of a connective is fixed by explaining what a proof is that involves the connective.
Basically, this explains the introduction rule for the connective.
By analysing proofs we can then also get

- consistency (from proof normalization and analysing normal derivations),
- a decision procedure (from the subformula property for normal derivations),
- completeness (w.r.t. Heyting algebra's).

Standard form for natural deduction rules

$$\frac{\Gamma \vdash \Phi_1 \quad \ldots \quad \Gamma \vdash \Phi_n \qquad \Gamma, \Psi_1 \vdash D \quad \ldots \quad \Gamma, \Psi_m \vdash D}{\Gamma \vdash D}$$

If the conclusion of a rule is $\Gamma \vdash D$, then the hypotheses of the rule can be of one of two forms:

1. $\Gamma, \Psi \vdash D$: we are given extra data Ψ to prove D from Γ. We call Ψ a Casus.
2. $\Gamma \vdash \Phi$: instead of proving D from Γ, we now need to prove Φ from Γ. We call Φ a Lemma.

One obvious advantage: we don't have to give the Γ explicitly, as it can be retrieved:

$$\frac{\vdash \Phi_1 \quad \ldots \quad \vdash \Phi_n \qquad \Psi_1 \vdash D \quad \ldots \quad \Psi_m \vdash D}{\vdash D}$$

Some well-known intuitionistics rules

Rules that follow this format:

$$\frac{\vdash A \vee B \qquad A \vdash D \qquad B \vdash D}{\vdash D}\,\text{el} \qquad \frac{\vdash A \wedge B \qquad A \vdash D}{\vdash D}\,\text{el}$$

$$\frac{\vdash A \qquad \vdash B}{\vdash A \wedge B}\,\text{in}$$

Rule that does not follow this format:

$$\frac{A \vdash B}{\vdash A \to B}\,\text{in}$$

Natural Deduction rules from truth tables

Let c be an n-ary connective c with truth table t_c and write
$\Phi = c(A_1, \ldots, A_n)$.
Each row of t_c gives rise to an elimination rule or an introduction rule for c.

$$\begin{array}{ccc|c} A_1 & \ldots & A_n & \Phi \\ \hline a_1 & \ldots & a_n & 0 \end{array} \quad \mapsto \quad \frac{\vdash \Phi \ldots \vdash A_j \text{ (if } a_j = 1) \ldots A_i \vdash D \text{ (if } a_i = 0) \ldots}{\vdash D}\,\text{el}$$

$$\begin{array}{ccc|c} A_1 & \ldots & A_n & \Phi \\ \hline b_1 & \ldots & b_n & 1 \end{array} \quad \mapsto \quad \frac{\ldots \vdash A_j \text{ (if } b_j = 1) \ldots A_i \vdash \Phi \text{ (if } b_i = 0) \ldots}{\vdash \Phi}\,\text{in}^i$$

$$\begin{array}{ccc|c} A_1 & \ldots & A_n & \Phi \\ \hline b_1 & \ldots & b_n & 1 \end{array} \quad \mapsto \quad \frac{\Phi \vdash D \ldots \vdash A_j \text{ (if } b_j = 1) \ldots A_i \vdash D \text{ (if } b_i = 0) \ldots}{\vdash D}\,\text{in}^c$$

Definition of the logics

Given a set of connectives $\mathcal{C} := \{c_1, \ldots, c_n\}$, we define the intuitionistic and classical natural deduction systems for $\mathcal{C}$, $\text{IPC}_\mathcal{C}$ and $\text{CPC}_\mathcal{C}$ as follows.

- Both $\text{IPC}_\mathcal{C}$ and $\text{CPC}_\mathcal{C}$ have an axiom rule

$$\frac{}{\Gamma \vdash A}\ \text{axiom(if } A \in \Gamma)$$

- Both $\text{IPC}_\mathcal{C}$ and $\text{CPC}_\mathcal{C}$ have the elimination rules for the connectives in $\mathcal{C}$.
- $\text{IPC}_\mathcal{C}$ has the intuitionistic introduction rules for the connectives in $\mathcal{C}$.
- $\text{CPC}_\mathcal{C}$ has the classical introduction rules for the connectives in $\mathcal{C}$.

Examples

Intuitionistic rules for $\wedge$: 3 elimination rules and one introduction rule:

$$\frac{\vdash A \wedge B \quad A \vdash D \quad B \vdash D}{\vdash D}\ \wedge\text{-el}_a \qquad \frac{\vdash A \wedge B \quad A \vdash D \quad \vdash B}{\vdash D}\ \wedge\text{-el}_b$$

$$\frac{\vdash A \wedge B \quad \vdash A \quad B \vdash D}{\vdash D}\ \wedge\text{-el}_c \qquad \frac{\vdash A \quad \vdash B}{\vdash A \wedge B}\ \wedge\text{-in}$$

- These rules can be shown to be equivalent to the well-known intuitionistic rules.
- These rules can be optimized and be reduced to 2 elimination rules and 1 introduction rule.

Examples

Rules for $\neg$: 1 elimination rule and 1 introduction rule.
Intuitionistic:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D}\,\neg\text{-el} \qquad \frac{A \vdash \neg A}{\vdash \neg A}\,\neg\text{-in}^i$$

Classical:

$$\frac{\neg A \vdash D \quad A \vdash D}{\vdash D}\,\neg\text{-in}^c$$

Example of a derivation

Using the classical rules for $\neg$, we show that $\neg\neg A \vdash A$ is derivable:

$$\dfrac{\dfrac{\neg\neg A, \neg A \vdash \neg\neg A \quad \neg\neg A, \neg A \vdash \neg A}{\neg\neg A, \neg A \vdash A}\,\neg\text{-el} \qquad \neg\neg A, A \vdash A}{\neg\neg A \vdash A}\,\neg\text{-in}^c$$

It can be proven that $\neg\neg A \vdash A$ is not derivable with the intuitionistic rules. As an example of the intuitionistic derivation rules we show that $A \vdash \neg\neg A$ is derivable:

$$\dfrac{\dfrac{A, \neg A \vdash \neg A \quad A, \neg A \vdash A}{A, \neg A \vdash \neg\neg A}\,\neg\text{-el}}{A \vdash \neg\neg A}\,\neg\text{-in}^i$$

Simplifying the set of rules

We can take a number of rules together and drop one or more hypotheses.
Example of $\wedge$:

$$\frac{\vdash A \wedge B \quad A \vdash D \quad B \vdash D}{\vdash D}\ \wedge\text{-el}_a \qquad \frac{\vdash A \wedge B \quad A \vdash D \quad \vdash B}{\vdash D}\ \wedge\text{-el}_b$$

$$\frac{\vdash A \wedge B \quad \vdash A \quad B \vdash D}{\vdash D}\ \wedge\text{-el}_c$$

These rules can be reduced to the following 2 equivalent $\wedge$-elim rules (that are also equivalent to the 2 standard $\wedge$-elim rules).

$$\frac{\vdash A \wedge B \quad A \vdash D}{\vdash D}\ \wedge\text{-el}_1 \qquad \frac{\vdash A \wedge B \quad B \vdash D}{\vdash D}\ \wedge\text{-el}_2$$

Lemma I to simplify the rules

$$\frac{\vdash \Phi_1 \quad \ldots \quad \vdash \Phi_n \quad \Psi_1 \vdash D \quad \ldots \quad \Psi_m \vdash D \quad A \vdash D}{\vdash D}$$

$$\frac{\vdash \Phi_1 \quad \ldots \quad \vdash \Phi_n \quad \vdash A \quad \Psi_1 \vdash D \quad \ldots \quad \Psi_m \vdash D}{\vdash D}$$

is equivalent to the system with these two rules replaced by

$$\frac{\vdash \Phi_1 \quad \ldots \quad \vdash \Phi_n \quad \Psi_1 \vdash D \quad \ldots \quad \Psi_m \vdash D}{\vdash D}$$

Lemma II to simplify the rules

We can replace a rule which has only one Casus by a rule where the Casus is the conclusion.

EXAMPLE: The rule $\wedge$-el$_1$ (left) can be replaced by the rule $\wedge$-el$'_1$ (right), which is the usual projection rule.

$$\frac{\vdash A \wedge B \quad A \vdash D}{\vdash D}\wedge\text{-el}_1 \qquad \frac{\vdash A \wedge B}{\vdash A}\wedge\text{-el}'_1$$

Lemma II to simplify the rules

A system with a derivation rule of the form to the left is equivalent to the system with this rule replaced by the rule on the right.

$$\frac{\vdash \Phi_1 \quad \ldots \quad \vdash \Phi_n \qquad \Psi \vdash D}{\vdash D} \qquad\qquad \frac{\vdash \Phi_1 \quad \ldots \quad \vdash \Phi_n}{\vdash \Psi}$$

The intuitionistic connectives

We have already seen the $\wedge, \neg$ rules. The optimised rules for $\vee, \rightarrow, \top$ and $\bot$ we obtain are:

$$\frac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D}\ \vee\text{-el} \qquad \frac{\vdash A}{\vdash A \vee B}\ \vee\text{-in}_1 \qquad \frac{\vdash B}{\vdash A \vee B}\ \vee\text{-in}_2$$

$$\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B} \rightarrow\text{-el} \qquad \frac{\vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_1 \qquad \frac{A \vdash A \rightarrow B}{\vdash A \rightarrow B} \rightarrow\text{-in}_2$$

$$\frac{}{\vdash \top}\ \top\text{-in} \qquad \frac{\vdash \bot}{\vdash D}\ \bot\text{-el}$$

The rules for the classical $\to$ connective

Derivation of Peirce's law:

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B} \to\text{-el} \qquad \frac{\vdash B}{\vdash A \to B} \to\text{-in}_1 \qquad \frac{A \vdash D \quad A \to B \vdash D}{\vdash D} \to\text{-in}_2^c$$

$$\dfrac{\dfrac{A \vdash A}{A \vdash ((A \to B) \to A) \to A} \qquad \dfrac{\dfrac{\dfrac{(A \to B) \to A \vdash (A \to B) \to A \quad A \to B \vdash A \to B}{A \to B, (A \to B) \to A \vdash A}}{A \to B, (A \to B) \to A \vdash ((A \to B) \to A) \to A}}{A \to B \vdash ((A \to B) \to A) \to A}}{\vdash ((A \to B) \to A) \to A} \to\text{-in}_2^c$$

Some observations

- For monotone connectives, the intuitionistic and classical rules are equivalent. (E.g. $\wedge$, $\vee$)
- For the non-monotonic connectives $\rightarrow$ and $\neg$, the classical intro rule for the one implies the classical intro rule for the other.
- Question: does that hold in general? (Conjecture: yes.)

The "If Then Else" connective

Notation: $A \to B / C$ for if A then B else C.

p	q	r	$p \to q/r$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

The optimized intuitionistic rules are:

$$\frac{\vdash A \to B/C \quad \vdash A}{\vdash B}\ \text{then-el} \qquad \frac{\vdash A \to B/C \quad A \vdash D \quad C \vdash D}{\vdash D}\ \text{else-el}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \to B/C}\ \text{then-in} \qquad \frac{A \vdash A \to B/C \quad \vdash C}{\vdash A \to B/C}\ \text{else-in}$$

The "If Then Else" connective is functionally complete

We define the usual intuitionistic connectives in terms of if-then-else, $\top$ and $\bot$:

$$A \dot{\vee} B \;:=\; A{\to}A/B \qquad A \dot{\wedge} B \;:=\; A{\to}B/A \qquad A \dot{\to} B \;:=\; A{\to}B/\top$$

LEMMA The defined connectives satisfy the original derivation rules for these same connectives.

COROLLARY The intuitionistic connective if-then-else, together with $\top$ and $\bot$, is functionally complete.

Kripke semantics for the intuitionistic rules

For each n-ary connective c, we assume a truth table $t_c : \{0,1\}^n \to \{0,1\}$ and the defined derivation rules.

DEFINITION A Kripke model is a triple $(W, \leq, \mathrm{at})$ where W is a set of worlds, $\leq$ a reflexive, transitive relation on W and a function $\mathrm{at} : W \to \wp(\mathrm{At})$ satisfying $w \leq w' \Rightarrow \mathrm{at}(w) \subseteq \mathrm{at}(w')$.

We define the notion φ is true in world w (usually written $w \Vdash \varphi$) by defining $[\![\varphi]\!]_w \in \{0,1\}$

DEFINITION of $[\![\varphi]\!]_w \in \{0,1\}$, by induction on φ:

- (atom) if φ is atomic, $[\![\varphi]\!]_w = 1$ iff $\varphi \in \mathrm{at}(w)$.
- (connective) for $\varphi = c(\varphi_1, \ldots, \varphi_n)$, $[\![\varphi]\!]_w = 1$ iff for each $w' \geq w$, $t_c([\![\varphi_1]\!]_{w'}, \ldots, [\![\varphi_1]\!]_{w'}) = 1$ where t_c is the truth table of c.

$\Gamma \models \psi$:= for each Kripke model and each world w, if $[\![\varphi]\!]_w = 1$ for each φ in Γ, then $[\![\psi]\!]_w = 1$.

Kripke semantics for the intuitionistic rules

LEMMA (Soundness) If $\Gamma \vdash \psi$, then $\Gamma \models \psi$
Proof. Induction on the derivation of $\Gamma \vdash \psi$.

For completeness we need to construct a special Kripke model.

- In the literature, the completeness of Kripke semantics is proved using *prime theories*.
- A theory is prime if it satisfies the disjunction property: if $\Gamma \vdash A \vee B$, then $\Gamma \vdash A$ or $\Gamma \vdash B$.
- We may not have $\vee$ in our set of connective, and we may have others that "behave $\vee$-like"',
- So we need to pass by the disjunction property for now.
- (Later, we will generalize the disjunction property to arbitrary n-ary intuitionistic connectives.)

Kripke semantics for the intuitionistic rules

DEFINITION For ψ a formula and Γ a set of formulas, we say that Γ is ψ-maximal if

- $\Gamma \nvdash \psi$ and
- for every formula $\varphi \notin \Gamma$ we have: $\Gamma, \varphi \vdash \psi$.

NB. Given ψ and Γ such that $\Gamma \nvdash \psi$, we can extend Γ to a ψ-maximal set Γ' that contains Γ.

Simple important facts about ψ-maximal sets Γ:

1. For every φ, we have $\varphi \in \Gamma$ or $\Gamma, \varphi \vdash \psi$.
2. For every φ, if $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.

Completeness of Kripke semantics

DEFINITION We define the Kripke model $U = (W, \leq, \mathsf{at})$:

- $W := \{(\Gamma, \psi) \mid \Gamma \text{ is a } \psi\text{-maximal set}\}$.
- $(\Gamma, \psi) \leq (\Gamma', \psi') := \Gamma \subseteq \Gamma'$.
- $\mathsf{at}(\Gamma, \psi) := \Gamma \cap \mathsf{At}$.

LEMMA In the model U we have, for all worlds $(\Gamma, \psi) \in W$:

$$\varphi \in \Gamma \iff [\![\varphi]\!]_{(\Gamma,\psi)} = 1 \quad (\forall \varphi)$$

Proof: Induction on the structure of φ.

THEOREM If $\Gamma \models \psi$, then $\Gamma \vdash \psi$.
Proof. Suppose $\Gamma \models \psi$ and $\Gamma \not\vdash \psi$. Then we can find a ψ-maximal superset Γ' of Γ such that $\Gamma' \not\vdash \psi$. In particular: ψ is not in Γ'. So (Γ', ψ) is a world in the Kripke model U in which each member of Γ is true, but φ is not. Contradiction, so $\Gamma \vdash \psi$.

A generalised disjunction property

We say that the n-ary connective c is i, j-splitting in case the truth table for c has the following shape

p_1	$\ldots$	p_i	$\ldots$	p_j	$\ldots$	p_n	$c(p_1, \ldots, p_n)$
–	$\ldots$	0	$\ldots$	0	$\ldots$	–	0
–	$\ldots$	0	$\ldots$	0	$\ldots$	–	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
–	$\ldots$	0	$\ldots$	0	$\ldots$	–	0
–	$\ldots$	0	$\ldots$	0	$\ldots$	–	0

In terms of t_c:

$$t_c(p_1, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_n) = 0$$

for all $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n \in \{0, 1\}$.

Property of splitting connectives

LEMMA Let c be an i,j-splitting connective and suppose $\vdash c(A_1,\ldots,A_n)$. Then $\vdash A_i$ or $\vdash A_j$.

Proof. Let $\varphi = c(A_1,\ldots,A_n)$ be a formula with $\vdash \varphi$. Suppose $\nvdash A_i$ and $\nvdash A_j$. Then there are Kripke models K_1 and K_2 such that $K_1 \nVdash A_i$ and $K_2 \nVdash A_j$. We construct a Kripke model K as the union of K_1 and K_2 where we add a special "root world" w_0 that is below all worlds of K_1 and K_2, with $\text{at}(w_0) = \emptyset$. We have

- $w_0 \nVdash A_i$, because $w_0 \leq w$ for some $w \in K_1$ with $w \nVdash A_i$;
- similarly $w_0 \nVdash A_j$. So, $[\![A_i]\!]_{w_0} = [\![A_j]\!]_{w_0} = 0$.
- But then $w_0 \nVdash \varphi$, because whatever the values of $[\![A_k]\!]_{w_0}$ are for $k \neq i,j$, $t_c([\![A_1]\!]_{w_0},\ldots,[\![A_n]\!]_{w_0}) = 0$.
- On the other hand, $w_0 \Vdash \varphi$, because $\vdash \varphi$, so: contradiction.

Examples of connectives with a splitting property

p	q	r	most(p, q, r)	$p \to q/r$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	0
1	1	0	1	1
1	1	1	1	1

- most is i, j-splitting for every i, j. Indeed, if $\vdash$ most(p, q, r), then $\vdash p$ or $\vdash q$, but also $\vdash p$ or $\vdash r$, and also $\vdash q$ or $\vdash r$.
- if-then-else is $1, 3$-splitting and $2, 3$-splitting (but not $1, 2$-splitting): if $\vdash p \to q/r$, then we have $\vdash p$ or $\vdash r$ and also $\vdash q$ or $\vdash r$.

Substituting a derivation in another

LEMMA: If $\Gamma \vdash \varphi$ and $\Delta, \varphi \vdash \psi$, then $\Gamma, \Delta \vdash \psi$

If Σ is a derivation of $\Gamma \vdash \varphi$ and Π is a derivation of $\Delta, \varphi \vdash \psi$, then we have the following derivation of $\Gamma, \Delta \vdash \psi$:

$$\begin{array}{ccc} \vdots\,\Sigma & & \vdots\,\Sigma \\ \Gamma \vdash \varphi & \ldots & \Gamma \vdash \varphi \\ & \vdots\,\Pi & \\ & \Delta \vdash \psi & \end{array}$$

In Π, every application of an (axiom) rule at a leaf, deriving $\Delta' \vdash \varphi$ for some $\Delta' \supseteq \Delta$ is replaced by a copy of a derivation Σ, which is also a derivation of $\Delta', \Gamma \vdash \varphi$.

Cuts in intuitionistic logic

An intuitionistic direct cut is a pattern of the following form, where $\Phi = c(A_1, \ldots, A_n)$. Remember these rules arise from rows in the truth table t_c:

p_1	$\ldots$	p_n	$c(p_1, \ldots, p_n)$
a_1	$\ldots$	a_n	0
b_1	$\ldots$	b_n	1

$$\dfrac{\dfrac{\ldots \quad \begin{array}{c}\vdots\,\Sigma_j\\ \Gamma \vdash A_j\end{array} \quad \ldots \qquad \ldots \quad \begin{array}{c}\vdots\,\Sigma_i\\ \Gamma, A_i \vdash \Phi\end{array} \quad \ldots}{\Gamma \vdash \Phi}\text{ in } \qquad \ldots \quad \begin{array}{c}\vdots\,\Pi_k\\ \Gamma \vdash A_k\end{array} \quad \ldots \qquad \ldots \quad \begin{array}{c}\vdots\,\Pi_\ell\\ \Gamma, A_\ell \vdash D\end{array} \quad \ldots}{\Gamma \vdash D}$$

- $b_j = 1$ for A_j and $b_i = 0$ for A_i
- $a_k = 1$ for A_k and $a_\ell = 0$ for A_ℓ

Eliminating a direct cut (I)

The *elimination of a direct cut* is defined by replacing the derivation pattern by another one. If $\ell = j$ (for some ℓ, j), replace

$$\dfrac{\dfrac{\ldots\ \begin{array}{c}\vdots\ \Sigma_j\\ \Gamma \vdash A_j\end{array}\ \ldots\quad \ldots\ \begin{array}{c}\vdots\ \Sigma_i\\ \Gamma, A_i \vdash \Phi\end{array}\ \ldots}{\Gamma \vdash \Phi}\qquad \ldots\ \begin{array}{c}\vdots\ \Pi_k\\ \Gamma \vdash A_k\end{array}\ \ldots\quad \ldots\ \begin{array}{c}\vdots\ \Pi_\ell\\ \Gamma, A_\ell \vdash D\end{array}\ \ldots}{\Gamma \vdash D}$$

by

$$\begin{array}{c}\begin{array}{c}\vdots\ \Sigma_j\\ \Gamma \vdash A_j\end{array}\ \ldots\ \begin{array}{c}\vdots\ \Sigma_j\\ \Gamma \vdash A_j\end{array}\\ \vdots\ \Pi_\ell\\ \Gamma \vdash D\end{array}$$

Eliminating a direct cut (II)

If $k = i$ (for some k, i), replace

$$\dfrac{\dfrac{\ldots\ \ \overset{\vdots\ \Sigma_j}{\Gamma \vdash A_j}\ \ \ldots\ \ \ldots\ \ \overset{\vdots\ \Sigma_i}{\Gamma, {\color{red}A_i} \vdash \Phi}\ \ \ldots}{\Gamma \vdash \Phi} \qquad \ldots\ \ \overset{\vdots\ \Pi_k}{\Gamma \vdash {\color{red}A_k}}\ \ \ldots\ \ \ldots\ \ \overset{\vdots\ \Pi_\ell}{\Gamma, A_\ell \vdash D}\ \ \ldots}{\Gamma \vdash D}$$

by

$$\dfrac{\overset{\overset{\overset{\vdots\ \Pi_k}{\Gamma \vdash A_i}\ \ \ldots\ \ \overset{\vdots\ \Pi_k}{\Gamma \vdash A_i}}{\vdots\ \Sigma_i}}{\Gamma \vdash \Phi} \qquad \ldots\ \ \overset{\vdots\ \Pi_k}{\Gamma \vdash A_i}\ \ \ldots\ \ \ldots\ \ \overset{\vdots\ \Pi_\ell}{\Gamma, A_\ell \vdash D}\ \ \ldots}{\Gamma \vdash D}$$

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Some observations

$$
\dfrac{\dfrac{\ldots \quad \begin{array}{c}\vdots\ \Sigma_j \\ \Gamma \vdash A_j\end{array} \quad \ldots \qquad \ldots \quad \begin{array}{c}\vdots\ \Sigma_i \\ \Gamma, A_i \vdash \Phi\end{array} \quad \ldots}{\Gamma \vdash \Phi} \qquad \ldots \quad \begin{array}{c}\vdots\ \Pi_k \\ \Gamma \vdash A_k\end{array} \quad \ldots \qquad \ldots \quad \begin{array}{c}\vdots\ \Pi_\ell \\ \Gamma, A_\ell \vdash D\end{array} \quad \ldots}{\Gamma \vdash D}
$$

- There may be several “matching” (k, i) pairs or matching (ℓ, j) pairs.
- So: cut-elimination is non-deterministic in general.

Cuts for if-then-else (I)

The cut-elimination rules for if-then-else are the following.

(then-then)

$$\dfrac{\dfrac{\Gamma \vdash A \qquad \begin{array}{c}\vdots\;\Sigma\\ \Gamma\vdash B\end{array}}{\Gamma \vdash A\to B/C}\,\text{in} \qquad \Gamma\vdash A}{\Gamma\vdash B}\,\text{el} \qquad \mapsto \qquad \begin{array}{c}\vdots\;\Sigma\\ \Gamma\vdash B\end{array}$$

(else-then)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\;\Sigma\\ \Gamma, A\vdash A\to B/C\end{array} \qquad \Gamma\vdash C}{\Gamma \vdash A\to B/C}\,\text{in} \qquad \begin{array}{c}\vdots\;\Pi\\ \Gamma\vdash A\end{array}}{\Gamma\vdash B}\,\text{el} \qquad \mapsto \qquad \dfrac{\begin{array}{c}\begin{array}{c}\vdots\;\Pi\\ \Gamma\vdash A\end{array}\;\ldots\;\begin{array}{c}\vdots\;\Pi\\ \Gamma\vdash A\end{array}\\ \vdots\;\Sigma\\ \Gamma\vdash A\to B/C\end{array} \qquad \begin{array}{c}\vdots\;\Pi\\ \Gamma\vdash A\end{array}}{\Gamma\vdash B}\,\text{el}$$

Cuts for if-then-else (II)

(then-else)

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\Sigma\\ \Gamma\vdash A\end{matrix}\qquad \Gamma\vdash B}{\Gamma\vdash A\to B/C}\,\text{in}\qquad \begin{matrix}\vdots\,\Pi\\ \Gamma,A\vdash D\end{matrix}\qquad \Gamma,C\vdash D}{\Gamma\vdash D}\,\text{el} \quad\mapsto\quad \begin{matrix}\begin{matrix}\vdots\,\Sigma\\ \Gamma\vdash A\end{matrix}\;\ldots\;\begin{matrix}\vdots\,\Sigma\\ \Gamma\vdash A\end{matrix}\\ \vdots\,\Pi\\ \Gamma\vdash D\end{matrix}$$

(else-else)

$$\dfrac{\dfrac{\Gamma,A\vdash A\to B/C\qquad \begin{matrix}\vdots\,\Sigma\\ \Gamma\vdash C\end{matrix}}{\Gamma\vdash A\to B/C}\,\text{in}\qquad \Gamma,A\vdash D\qquad \begin{matrix}\vdots\,\Pi\\ \Gamma,C\vdash D\end{matrix}}{\Gamma\vdash D}\,\text{el} \quad\mapsto\quad \begin{matrix}\begin{matrix}\vdots\,\Sigma\\ \Gamma\vdash C\end{matrix}\;\ldots\;\begin{matrix}\vdots\,\Sigma\\ \Gamma\vdash C\end{matrix}\\ \vdots\,\Pi\\ \Gamma\vdash D\end{matrix}$$

Curry-Howard proofs-as-terms

We define rules for the judgment $\Gamma \vdash t : A$, where

- A is a formula,
- Γ is a set of declarations $\{x_1 : A_1, \ldots, x_m : A_m\}$, where the A_i are formulas and the x_i are term-variables,
- t is a proof-term.

For a connective $c \in \mathcal{C}$ of arity n, we have an introduction term $\iota(t_1, \ldots, t_n)$ and an elimination term $\varepsilon(t_0, t_1, \ldots, t_n)$, where the t_i are again proof-terms or of the shape $\lambda x.t'$, where x is a term-variable and t' is a proof-term.

Curry-Howard typing rules

The typing rules are:

$$\frac{}{\Gamma \vdash x_i : A_i}\; x_i : A_i \in \Gamma$$

$$\frac{\ldots \Gamma \vdash t_j : A_j \ldots \quad \ldots \Gamma, y_i : A_i \vdash q_i : \Phi \ldots}{\Gamma \vdash \iota(\overrightarrow{t}, \overrightarrow{\lambda y.q}) : \Phi \ldots}\; \text{in}$$

$$\frac{\Gamma \vdash t_0 : \Phi \ldots \quad \ldots \Gamma \vdash s_k : A_k \ldots \quad \ldots \Gamma, z_\ell : A_\ell \vdash r_\ell : D \ldots}{\Gamma \vdash \varepsilon(t_0, \overrightarrow{s}, \overrightarrow{\lambda z.r}) : D}\; \text{el}$$

In the in-rule, $\overrightarrow{t}$ is the sequence of terms $t_1, \ldots, t_p$ for the 1-entries and $\overrightarrow{\lambda y.q}$ is the sequence of terms $\lambda y_1.q_1, \ldots, \lambda y_r.q_r$ for all the 0-entries in the truth table. (Similar for $\overrightarrow{s}$ and $\overrightarrow{\lambda z.r}$.)

Reductions on terms

Term reduction rules that correspond to the elimination of direct cuts.

- Given a direct cut as defined before, we add reduction rules for the associated terms as.
- For simplicity of presentation we write the "matching cases" as last term of the sequence. (NB! In general there are multiple matching cases!)

For the $\ell = j$ case:

$$\varepsilon(\iota(\vec{t}, t_j, \overrightarrow{\lambda y.q}), \vec{s}, \overrightarrow{\lambda z.r}, \lambda z_\ell.r_\ell) \quad \longrightarrow \quad r_\ell[z_\ell := t_j]$$

For the $k = i$ case:

$$\varepsilon(\iota(\vec{t}, \overrightarrow{\lambda y.q}, \lambda y_i.q_i), \vec{s}, s_k, \overrightarrow{\lambda z.r}) \quad \longrightarrow \quad \varepsilon(q_i[y_i := s_k], \vec{s}, s_k, \overrightarrow{\lambda z.r})$$

Optimized reductions on terms

- The definition gives a reduction rule for every combination of an elimination and an introduction.
- For an n-ary connective, there are 2^n constructors (intro plus elim constructors).
- Usually, we want to just look at the optimized rules
- For these optimized rules, there is also a straightforward definition of proof-terms and of the reduction relation associated with cut-elimination.
- The proof-terms for the optimized rules can be defined in terms of the terms for the full calculus, and the reduction rules for the optimized proof terms are an instance of reductions in the full calculus (often multi-step).

The calculus λif-then-else

DEFINITION We define the calculus λif-then-else as a calculus for terms and reductions for the if-then-else logic (Γ omitted):

$$\frac{\vdash t_0 : A \to B/C \quad \vdash a : A}{\vdash \varepsilon_1(t_0, a) : B}\ \text{then-el} \qquad \frac{\vdash t_0 : A \to B/C \quad x : A \vdash t : D \quad y : C \vdash q : D}{\vdash \varepsilon_2(t_0, \lambda x.t, \lambda y.q) : D}\ \text{else-el}$$

$$\frac{\vdash a : A \quad \vdash b : B}{\vdash \iota_1(a, b) : A \to B/C}\ \text{then-in} \qquad \frac{x : A \vdash t : A \to B/C \quad \vdash c : C}{\vdash \iota_2(\lambda x.t, c) : A \to B/C}\ \text{else-in}$$

The reduction rules are

$$\begin{array}{lcl}
\varepsilon_1(\iota_1(a, b), a') & \longrightarrow & b \\
\varepsilon_1(\iota_2(\lambda x.t, c), a) & \longrightarrow & \varepsilon_1(t[x := a], a) \\
\varepsilon_2(\iota_1(a, b), \lambda x.t, \lambda y.q) & \longrightarrow & t[x := a] \\
\varepsilon_2(\iota_2(\lambda x.t, c), \lambda z.d, \lambda y.q) & \longrightarrow & q[y := c]
\end{array}$$

Strong Normalization for λif-then-else

We prove Strong Normalization for the reductions in λif-then-else by adapting the saturated sets method.

But . . . what we would really want is that proof-terms in normal form have the subformula property: if $t : A$, then the type of a sub-term of t is a sub-type of A.

Then we can derive

- consistency of the logic
- decidability of the logic
- and thereby a (simple?) decision procedure for full IPC.

We need to add permuting reduction rules

$$\varepsilon_1(\varepsilon_2(t_0, \lambda x.t, \lambda y.q), e) \longrightarrow \varepsilon_2(t_0, \lambda x.\varepsilon_1(t, e), \lambda y.\varepsilon_1(q, e))$$

$$\varepsilon_2(\varepsilon_2(t_0, \lambda x.t, \lambda y.q), \lambda v.r, \lambda z.s) \longrightarrow \varepsilon_2(t_0, \lambda x.\varepsilon_2(t, \lambda v.r, \lambda z.s), \lambda y.\varepsilon_2(q, \lambda v.r, \lambda z.s)$$

Conclusions, Further work, Related work

Conclusions

- Simple way to construct derivation system for new connectives, intuitionistically and classically
- Study connectives "in isolation". (Without defining them.)
- Generic Kripke semantics

Some open questions/ further work:

- Constructive Kripke semantics completeness
- Meaning of the new connectives as data types
- General definition of classical cut-elimination
- Relation with other term calculi for classical logic: subtraction logic, $\lambda\mu$ (Parigot), $\bar{\lambda}\mu\tilde{\mu}$ (Curien, Herbelin).
- SN for λif-then-else with permuting cuts

Related work:

- Jan von Plato and Sara Negri
- Peter Schroeder-Heister

Equivalences of if-then-else

```
Inductive ite : Prop :=
| then_in : A -> B -> ite
| else_in : (A->ite) -> C -> ite.

ite_ind : forall P : Prop, (A -> B -> P) ->
    ((A -> ite) -> (A -> P) -> C -> P) -> ite -> P

Lemma then_el : ite -> A -> B.
Lemma else_el : forall D:Prop, ite -> (A->D) ->
    (C->D) -> D.
```

Equivalent to

- Higher order definition
  $\forall D : \texttt{Prop}, (A \to B \to D) \to ((A \to D) \to C \to D) \to D.$
- $(A \to B) \wedge (A \vee C)$
- Axiomatically assuming `then_in`, `else_in`, `then_el` and `else_el`