Deriving derivation rules from truth tables: classically, constructively and proof reduction

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World Logic Day 2021
Tallinn Estonia
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Classically, the meaning of a propositional connective is fixed by its truth table. This immediately implies

- consistency,
- a decision procedure,
- completeness (w.r.t. Boolean algebra’s).

Constructively (following the Brouwer-Heyting-Kolmogorov interpretation), the meaning of a connective is fixed by explaining what a proof is that involves the connective. Basically, this explains the introduction rule(s) for each connective, from which the elimination rules follow (Prawitz).

By analysing constructive proofs we then also get

- consistency (from proof normalization),
- a decision procedure (from the subformula property),
- completeness (w.r.t. Heyting algebra’s and Kripke models).
This talk

• Derive natural deduction rules for a connective from its truth table definition.
  • Also works for constructive logic.
  • Gives natural deduction rules for a connective “in isolation”
  • Also gives (constructive) rules for connectives that haven’t been studied so far, like if-then-else and nand.

• General definition, both the constructive and the classical case.

• Relation to “standard” natural deduction rules and known connectives.

• General Kripke model for the constructive connectives. (Sound and Complete)

• Curry-Howard proofs-as-terms interpretation for derivations and normalization of proof-reduction

• Interpreting classical proofs as terms.
Standard form for natural deduction rules

\[ \Gamma \vdash A_1 \ldots \Gamma \vdash A_n \quad \Gamma, B_1 \vdash D \quad \ldots \quad \Gamma, B_m \vdash D \]

\[ \Gamma \vdash D \]

If the conclusion of a rule is \( \Gamma \vdash D \), then the hypotheses of the rule can be of one of two forms:

1. \( \Gamma \vdash A \): instead of proving \( D \) from \( \Gamma \), we now need to prove \( A \) from \( \Gamma \). We call \( A \) a **Lemma**.

2. \( \Gamma, B \vdash D \): we are given extra data \( B \) to prove \( D \) from \( \Gamma \). We call \( B \) a **Casus**.

We don’t give the \( \Gamma \) explicitly (it can be retrieved):

\[ \vdash A_1 \ldots \vdash A_n \quad B_1 \vdash D \quad \ldots \quad B_m \vdash D \]

\[ \vdash D \]
Some well-known constructive rules

Rules that follow this format:

\[
\begin{align*}
\vdash A \lor B & \quad A \vdash D & \quad B \vdash D \\
\hline \\
\vdash D & \quad \lor\text{-el}
\end{align*}
\]

\[
\begin{align*}
\vdash A \land B & \quad A \vdash D \\
\hline \\
\vdash D & \quad \land\text{-el}
\end{align*}
\]

\[
\begin{align*}
\vdash A & \quad \vdash B \\
\hline \\
\vdash A \land B & \quad \land\text{-in}
\end{align*}
\]

Rule that does not follow this format:

\[
\begin{align*}
A \vdash B \\
\hline \\
\vdash A \rightarrow B & \quad \rightarrow\text{-in}
\end{align*}
\]
Let $c$ be an $n$-ary connective $c$ with truth table $t_c$. Each row of $t_c$ gives rise to an elimination rule or an introduction rule for $c$. (We write $\Phi = c(A_1, \ldots, A_n).$)

\[
\begin{array}{ccc}
A_1 & \ldots & A_n \\
p_1 & \ldots & p_n
\end{array}
\begin{array}{c}
\Phi \\
0
\end{array} \quad \Rightarrow
\begin{array}{c}
\vdash \Phi \ldots \vdash A_j \text{ (if } p_j = 1) \ldots A_i \vdash D \text{ (if } p_i = 0) \\
\vdash D
\end{array}
\]

\text{constructive intro}

\[
\begin{array}{ccc}
A_1 & \ldots & A_n \\
q_1 & \ldots & q_n
\end{array}
\begin{array}{c}
\Phi \\
1
\end{array} \quad \Rightarrow
\begin{array}{c}
\vdash A_j \text{ (if } q_j = 1) \ldots A_i \vdash \Phi \text{ (if } q_i = 0) \\
\vdash \Phi
\end{array}
\]

\text{classical intro}

\[
\begin{array}{ccc}
A_1 & \ldots & A_n \\
r_1 & \ldots & r_n
\end{array}
\begin{array}{c}
\Phi \\
1
\end{array} \quad \Rightarrow
\begin{array}{c}
\Phi \vdash D \ldots \vdash A_j \text{ (if } r_j = 1) \ldots A_i \vdash D \text{ (if } r_i = 0) \\
\vdash D
\end{array}
\]
Examples

Constructive rules for $\land$ (3 elim rules and one intro rule):

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$A \land B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
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<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
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<td>1</td>
</tr>
</tbody>
</table>

$\vdash A \land B \quad A \vdash D \quad B \vdash D$  

$\vdash D \quad \land\text{-el}_{00}$

$\vdash A \land B \quad \vdash A \quad B \vdash D$  

$\vdash D \quad \land\text{-el}_{10}$

$\vdash A \land B \quad A \vdash D$  

$\vdash B \quad \land\text{-el}_{01}$

$\vdash A \land B \quad \vdash A$  

$\vdash B \quad \land\text{-in}_{11}$

- Can be shown to be equivalent to the well-known constructive rules.
- These rules can be optimized to 3 rules.
Examples

Rules for $\neg$: 1 elimination rule and 1 introduction rule.

\[
\begin{array}{c|c}
A & \neg A \\
0 & 1 \\
1 & 0 \\
\end{array}
\]

Constructive:

\[
\frac{\vdash \neg A \quad \vdash A}{\vdash D} \quad \neg\text{-el} \quad \frac{A \vdash \neg A}{\vdash \neg A} \quad \neg\text{-in}^i
\]

Classical:

\[
\frac{\vdash \neg A \quad \vdash A}{\vdash D} \quad \neg\text{-el} \quad \frac{\neg A \vdash D \quad A \vdash D}{\vdash D} \quad \neg\text{-in}^c
\]
Lemma 1 to simplify the rules

\[ \frac{\vdash A_1 \ldots \vdash A_n \quad B_1 \vdash D \ldots B_m \vdash D \quad C \vdash D}{\vdash D} \]

\[ \frac{\vdash A_1 \ldots \vdash A_n \quad \vdash C \quad B_1 \vdash D \ldots B_m \vdash D}{\vdash D} \]

is equivalent to the system with these two rules replaced by

\[ \frac{\vdash A_1 \ldots \vdash A_n \quad B_1 \vdash D \ldots B_m \vdash D}{\vdash D} \]
A system with a deduction rule of the form to the left is equivalent to the system with this rule replaced by the rule on the right.

\[
\begin{array}{c}
\vdash A_1 \ldots \vdash A_n \\
B \vdash D
\end{array}
\quad \vdash A_1 \ldots \vdash A_n
\quad \begin{array}{c}
\vdash D \\
\vdash B
\end{array}
\]
We have already seen the $\land$, $\neg$ rules. The optimized rules for $\lor$, $\rightarrow$, $\top$ and $\bot$ we obtain are:

\[
\begin{align*}
\frac{\vdash A \lor B \quad A \vdash D \quad B \vdash D}{\vdash D} & \quad \lor\text{-el} \\
\frac{\vdash A \quad \vdash A \lor B}{\vdash A \lor B} & \quad \lor\text{-in}_1 \\
\frac{\vdash B \quad \vdash A \lor B}{\vdash A \lor B} & \quad \lor\text{-in}_2 \\
\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B} & \quad \rightarrow\text{-el} \\
\frac{\vdash B \quad \vdash A \rightarrow B}{\vdash A \rightarrow B} & \quad \rightarrow\text{-in}_1 \\
\frac{\vdash A \rightarrow B \quad \vdash A \rightarrow B}{\vdash A \rightarrow B} & \quad \rightarrow\text{-in}_2 \\
\frac{\vdash \top}{\vdash \top} & \quad \top\text{-in} \\
\frac{\vdash \bot}{\vdash \bot} & \quad \bot\text{-el} \\
\end{align*}
\]
The rules for the classical $\rightarrow$ connective

$$
\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B} \quad \rightarrow\text{-el}
$$

$$
\frac{\vdash B}{\vdash A \rightarrow B} \quad \rightarrow\text{-in}_1
$$

$$
A \rightarrow B \vdash D \quad A \vdash D \\
\frac{\vdash D}{\vdash A \rightarrow B} \quad \rightarrow\text{-in}^{\mathcal{C}}_2
$$

Derivation of Peirce’s law:

$$
\frac{(A \rightarrow B) \rightarrow A \vdash (A \rightarrow B) \rightarrow A \quad A \rightarrow B \vdash A \rightarrow B}{A \rightarrow B, (A \rightarrow B) \rightarrow A \vdash A}
$$

$$
\frac{A \vdash A}{A \rightarrow B, (A \rightarrow B) \rightarrow A \vdash ((A \rightarrow B) \rightarrow A) \rightarrow A}
$$

$$
\frac{A \rightarrow B \vdash ((A \rightarrow B) \rightarrow A) \rightarrow A}{\vdash ((A \rightarrow B) \rightarrow A) \rightarrow A}
$$
The “If Then Else” connective

Notation: $A \rightarrow B/C$ for if $A$ then $B$ else $C$.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>$p \rightarrow q/r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>1</td>
</tr>
</tbody>
</table>

The optimized constructive rules are:

- $\vdash A \rightarrow B/C \quad \vdash A$  \quad \text{then-el}  \quad \vdash B$
- $\vdash A \quad \vdash B$  \quad \text{then-in}  \quad \vdash A \rightarrow B/C$
- $\vdash A \rightarrow B/C \quad A \vdash D \quad C \vdash D$  \quad \text{else-el}  \quad \vdash D$
- $A \vdash A \rightarrow B/C \quad \vdash C$  \quad \text{else-in}  \quad \vdash A \rightarrow B/C$
Some facts about constructive “If Then Else”

\[ A \rightarrow B / C \text{ is logically equivalent to } (A \rightarrow B) \land (A \lor C) \]

We have the well-known classical equivalence

\[
\text{if } A \text{ then } B \text{ else } B \equiv B
\]

We don’t have the other well-known classical equivalences

\[
\text{if } (\text{if } A \text{ then } B \text{ else } C) \text{ then } D \text{ else } E \not\vdash
\]

\[
\text{if } A \text{ then } (\text{if } B \text{ then } D \text{ else } E) \text{ else } (\text{if } C \text{ then } D \text{ else } E)
\]

\[
\text{if } A \text{ then } (\text{if } B \text{ then } D \text{ else } E) \text{ else } (\text{if } C \text{ then } D \text{ else } E) \not\vdash
\]

\[
\text{if } (\text{if } A \text{ then } B \text{ else } C) \text{ then } D \text{ else } E
\]
We can define the usual constructive connectives in terms of if-then-else, $\top$ and $\bot$:

$$A \lor B := A \to A / B \quad A \land B := A \to B / A$$

$$A \to B := A \to B / \top \quad \neg A := A \to \bot / \top$$

**Lemma** The defined connectives satisfy the original constructive deduction rules for these same connectives.

**Corollary** The constructive connective if-then-else, together with $\top$ and $\bot$, is functionally complete.
The truth table for \( \text{nand}(A, B) \), which we write as \( A \uparrow B \) is as follows.

\[
\begin{array}{ccc}
A & B & A \uparrow B \\
0 & 0 & 1 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\]

From this we derive the following optimized rules.

\[
\frac{A \vdash A \uparrow B}{\vdash A \uparrow B} \quad \uparrow\text{-inl} \quad \frac{B \vdash A \uparrow B}{\vdash A \uparrow B} \quad \uparrow\text{-inr} \quad \frac{\vdash A \uparrow B \vdash A \vdash B}{\vdash D} \quad \uparrow\text{-el}
\]
The usual connectives can be defined in terms of nand.

\[\begin{align*}
\neg A & := A \uparrow A \\
A \lor B & := (A \uparrow A) \uparrow (B \uparrow B) \\
A \land B & := (A \uparrow B) \uparrow (A \uparrow B) \\
A \rightarrow B & := A \uparrow (B \uparrow B)
\end{align*}\]

This gives rise to an embedding \((-)\) of intuitionistic proposition logic \(\vdash_i\) into the nand-logic \(\vdash\).

**Proposition** For \(A\) a formula in proposition logic,

\[\vdash_i \neg \neg A \iff \vdash (A)\uparrow.\]
Kripke semantics for the constructive rules

For each $n$-ary connective $c$, we assume a truth table $t_c : \{0, 1\}^n \rightarrow \{0, 1\}$ and the defined constructive deduction rules.

**Definition** A Kripke model is a triple $(W, \leq, at)$ where $W$ is a set of worlds, $\leq$ a reflexive, transitive relation on $W$ and a function $at : W \rightarrow \wp(\text{At})$ satisfying $w \leq w' \Rightarrow at(w) \subseteq at(w')$.

We define the notion $\varphi$ is true in world $w$ (usually written $w \models \varphi$) by defining $\llbracket \varphi \rrbracket_w \in \{0, 1\}$

**Definition** of $\llbracket \varphi \rrbracket_w \in \{0, 1\}$, by induction on $\varphi$:
- (atom) if $\varphi$ is atomic, $\llbracket \varphi \rrbracket_w = 1$ iff $\varphi \in \text{at}(w)$.
- (connective) for $\varphi = c(\varphi_1, \ldots, \varphi_n)$, $\llbracket \varphi \rrbracket_w = 1$ iff for each $w' \geq w$, $t_c(\llbracket \varphi_1 \rrbracket_{w'}, \ldots, \llbracket \varphi_n \rrbracket_{w'}) = 1$ where $t_c$ is the truth table of $c$.

$\Gamma \models \psi :=$ for each Kripke model and each world $w$, if $\llbracket \varphi \rrbracket_w = 1$ for each $\varphi$ in $\Gamma$, then $\llbracket \psi \rrbracket_w = 1$. 
**Lemma** (Soundness) If $\Gamma \vdash \psi$, then $\Gamma \models \psi$

Proof. Induction on the derivation of $\Gamma \vdash \psi$.

For completeness we need to construct a special Kripke model.

- In the literature, the completeness of Kripke semantics is proved using *prime theories*.
- A theory is prime if it satisfies the *disjunction property*: if $\Gamma \vdash A \lor B$, then $\Gamma \vdash A$ or $\Gamma \vdash B$.
- We may not have $\lor$ in our set of connective, and we may have others that “behave $\lor$-like”,
- (But we can generalize the disjunction property to arbitrary $n$-ary constructive connectives that are “splitting”.)
- We apply a kind of Lindenbaum construction (also used by Milne for the classical case).
**Definition** For $\psi$ a formula and $\Gamma$ a set of formulas, we say that $\Gamma$ is $\psi$-maximal if

- $\Gamma \not\vdash \psi$ and
- for every formula $\varphi \notin \Gamma$ we have: $\Gamma, \varphi \vdash \psi$.

NB. Given $\psi$ and $\Gamma$ such that $\Gamma \not\vdash \psi$, we can extend $\Gamma$ to a $\psi$-maximal set $\Gamma'$ that contains $\Gamma$.

Simple important facts about $\psi$-maximal sets $\Gamma$:

1. For every $\varphi$, we have $\varphi \in \Gamma$ or $\Gamma, \varphi \vdash \psi$.
2. For every $\varphi$, if $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$. 
Completeness of Kripke semantics

**Definition** We define the Kripke model $U = (\mathcal{W}, \leq, \text{at})$:

- $\mathcal{W} := \{ (\Gamma, \psi) \mid \Gamma \text{ is a } \psi\text{-maximal set} \}$.
- $(\Gamma, \psi) \leq (\Gamma', \psi') := \Gamma \subseteq \Gamma'$.
- $\text{at}(\Gamma, \psi) := \Gamma \cap \text{At}$.

**Lemma** In the model $U$ we have, for all worlds $(\Gamma, \psi) \in \mathcal{W}$:

$$\varphi \in \Gamma \iff \llbracket \varphi \rrbracket_{(\Gamma, \psi)} = 1 \quad (\forall \varphi)$$

Proof: Induction on the structure of $\varphi$.

**Theorem** If $\Gamma \models \psi$, then $\Gamma \vdash \psi$.

Proof. Suppose $\Gamma \models \psi$ and $\Gamma \not\vdash \psi$. Then we can find a $\psi$-maximal superset $\Gamma'$ of $\Gamma$ such that $\Gamma' \not\models \psi$. In particular: $\psi$ is not in $\Gamma'$. So $(\Gamma', \psi)$ is a world in the Kripke model $U$ in which each member of $\Gamma$ is true, but $\psi$ is not. Contradiction to $\Gamma \models \psi$, so $\Gamma \vdash \psi$. 
The $n$-ary connective $c$ is $i,j$-splitting in case
\[
t_c(p_1, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_n) = 0
\]
for all $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n \in \{0, 1\}$.

**Lemma** For $c$ an $i,j$-splitting connective, if $\vdash c(A_1, \ldots, A_n)$, then $\vdash A_i$ or $\vdash A_j$.

For example: if $\vdash A \rightarrow B / C$, then $\vdash A$ or $\vdash C$. (And also: if $\vdash A \rightarrow B / C$, then $\vdash B$ or $\vdash C$.)

An $n$-ary connective $c$ is **monotonic** if $t_c : \{0, 1\}^n \rightarrow \{0, 1\}$ is monotonic under the ordering induced by $0 \leq 1$.

**Lemma** For $c$ monotonic, the classical and constructive derivation rules are equivalent.

**Lemma** For $c_1, c_2$ non-monotonic, if we take the classical rules for $c_1$ and the constructive rules for $c_2$, we can derive the classical rules for $c_2$. 
Lemma: If $\Gamma \vdash A$ and $\Delta, A \vdash B$, then $\Gamma, \Delta \vdash B$

If $\Sigma$ is a deduction of $\Gamma \vdash A$ and $\Pi$ is a deduction of $\Delta, A \vdash B$, then we have the following deduction of $\Gamma, \Delta \vdash B$:

\[
\begin{array}{c}
\Sigma \\
\Gamma \vdash A \quad \ldots \quad \Gamma \vdash A \\
\Sigma \\
\Pi \\
\Delta \vdash B
\end{array}
\]

In $\Pi$, every application of an (axiom) rule at a leaf, deriving $\Delta' \vdash A$ for some $\Delta' \supseteq \Delta$ is replaced by a copy of a deduction $\Sigma$, which is also a deduction of $\Delta', \Gamma \vdash A$. 
Detours (cuts) in constructive logic

Remember that the rules for $c$ arise from rows in the truth table $t_c$:

<table>
<thead>
<tr>
<th>$A_1$</th>
<th>...</th>
<th>$A_n$</th>
<th>$c(A_1, \ldots, A_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>...</td>
<td>$p_n$</td>
<td>0</td>
</tr>
<tr>
<td>$q_1$</td>
<td>...</td>
<td>$q_n$</td>
<td>1</td>
</tr>
</tbody>
</table>

**Definition** A *detour convertibility* is a pattern of the following form, where $\Phi = c(A_1, \ldots, A_n)$.

\[ \Gamma \vdash A_j \quad \Gamma, A_i \vdash \Phi \]

\[ \Gamma \vdash \Phi \quad \Gamma \vdash A_k \quad \Gamma, A_\ell \vdash D \]

\[ \Gamma \vdash D \]

- $q_j = 1$ for $A_j$ and $q_i = 0$ for $A_i$
- $p_k = 1$ for $A_k$ and $p_\ell = 0$ for $A_\ell$
The *elimination of a detour* is defined by replacing the deduction pattern by another one. If $j = \ell$ (for some $j, \ell$, so $A_j = A_\ell$), replace

\[
\ldots \quad \Sigma_j \quad \ldots \quad \Gamma \vdash A_j \quad \ldots \quad \Sigma_j \quad \ldots
\in
\Gamma \vdash \Phi
\]

by

\[
\ldots \quad \Pi_k \quad \ldots \quad \Gamma \vdash A_k \quad \ldots \quad \Pi_\ell \quad \ldots
\]\n
\[
\Gamma \vdash D
\]
If \( i = k \) (for some \( i, k \), so \( A_i = A_k \)), replace

\[ \Sigma_j \quad \Gamma \vdash A_j \quad \ldots \quad \Gamma, A_i \vdash \Phi \quad \ldots \quad \Sigma_i \]

\[ \Gamma \vdash \Phi \]

\[ \Pi_k \quad \Gamma \vdash A_k \quad \ldots \quad \Gamma, A_\ell \vdash D \quad \ldots \quad \Pi_\ell \]

\[ \Pi_k \quad \Gamma \vdash A_k \quad \ldots \quad \Gamma \vdash D \quad \Pi_\ell \]

by

\[ \Pi_k \quad \Gamma \vdash A_k \quad \ldots \quad \Gamma \vdash \Phi \quad \ldots \quad \Pi_k \]

\[ \Gamma \vdash D \quad \Pi_\ell \]

\[ \Gamma \vdash D \quad \Pi_\ell \]

\[ \Gamma \vdash D \quad \Pi_\ell \]
There can be several “matching” \((i, k)\) or \((j, \ell)\) pairs.

So: detour conversion ("\(\beta\)-rule") is non-deterministic in general.
Permutation convertibility: Definition

Let $c$ and $c'$ be connectives of arity $n$ and $n'$, with elimination rules $r$ and $r'$ respectively. A permutation convertibility in a derivation is a pattern of the following form, where $\Phi = c(B_1, \ldots, B_n)$, $\Psi = c'(A_1, \ldots, A_{n'})$.

\[
\begin{array}{cccc}
\vdash \Psi & \vdash A_j & \ldots & \vdash A_i & \vdash \Phi & \ldots \\
\Sigma_j & & \Sigma_i & \text{el}_{r'} & & \text{el}_{r'} \\
\end{array}
\]

\[
\vdash \Phi \ldots \vdash B_k & \ldots \vdash B_\ell & \vdash D & \ldots \\
\Pi_k & & \Pi_\ell & \text{el}_{r} & & \text{el}_{r} \\
\vdash D
\]

- $A_j$ ranges over all propositions that have a 1 in the truth table of $c'$; $A_i$ ranges over all propositions that have a 0,
- $B_k$ ranges over all propositions that have a 1 in the truth table of $c$; $B_\ell$ ranges over all propositions that have a 0.
The permutation conversion is defined by replacing the derivation pattern on the previous slide by

\[
\frac{
\frac{
\frac{
\sum_j}{A_i \vdash \Phi}
\ldots
\frac{
\Pi_k}{A_i \vdash B_k}
\ldots
\frac{
\Pi_\ell}{A_i, B_\ell \vdash D}
\ldots
}{\sum_j}{el_r
}{\vdash D}
}{el_{r'}}
\]

This gives rise to copying of sub-derivations: for every \( A_i \) we copy the sub-derivations \( \Pi_1, \ldots, \Pi_n \).
We define rules for the judgment $\Gamma \vdash t : A$, where

- $A$ is a formula,
- $\Gamma$ is a set of declarations $\{x_1 : A_1, \ldots, x_m : A_m\}$, where the $A_i$ are formulas and the $x_i$ are term-variables,
- $t$ is a proof-term:

\[
  t ::= x \mid \{t ; \lambda x : A.t\}_r \mid t \cdot_r [t ; \lambda x : A.t]
\]

where $x$ ranges over variables and $r$ ranges over the rules.

For a connective $c \in C$, $r$ an introduction rule for $c$ and $r'$ an elimination rule for $c$, we have

- an **introduction term** $\{t ; \lambda x : A.t\}_r$
- an **elimination term** $t \cdot_{r'} [t ; \lambda x : A.t]$
Let $\Phi = c(A_1, \ldots, A_n)$ and $r$ a rule for $c$.

\[
\frac{}{\Gamma \vdash x_i : A_i} \quad \text{if } x_i : A_i \in \Gamma
\]
\[
\frac{}{\Gamma \vdash p_j : A_j \ldots \Gamma, y_i : A_i \vdash q_i : \Phi \ldots} {\text{in}}
\]
\[
\frac{}{\Gamma \vdash \{\overline{p} ; \overline{\lambda y : A.q}\}_r : \Phi}
\]
\[
\frac{}{\Gamma \vdash t : \Phi \ldots \Gamma \vdash p_k : A_k \ldots \Gamma, y_\ell : A_\ell \vdash q_\ell : D} \quad \text{el}
\]
\[
\frac{}{\Gamma \vdash t \cdot_r [\overline{p} ; \overline{\lambda y : A.q}] : D}
\]

Here, $\overline{p}$ is the sequence of terms $p_1, \ldots, p_m'$ for all the 1-entries in rule $r$ of the truth table, and $\overline{\lambda y : A.q}$ is the sequence of terms $\lambda y_1 : A_1.q_1, \ldots, \lambda y_m : A_m.q_m$ for all the 0-entries in $r$. 
Reductions on terms for detours

Term reduction rules that correspond to detour conversions.

- For simplicity we write the “matching cases” as last term of the sequence.
- For the $j = \ell$ case, that is, $p_j : A_j$ and $y_\ell : A_\ell$ with $A_j = A_\ell$:
  \[
  \{p, p_j ; \lambda x. q\} \cdot [s ; \lambda y.r, \lambda y_\ell.r_\ell] \rightarrow_a r_\ell[y_\ell := p_j]
  \]

- For the $i = k$ case, that is, $x_i : A_i$ and $s_k : A_k$ with $A_i = A_k$:
  \[
  \{p ; \lambda x.q, \lambda x_i.q_i\} \cdot [s, s_k ; \lambda y.r] \rightarrow_a q_i[x_i := s_k] \cdot [s, s_k ; \lambda y.r]
  \]

$p, p_j$ should be understood as a sequence $p_1, \ldots, p_j, \ldots p_m'$, where the $p_j$ that matches the $r_\ell$ in $\lambda y.r, \lambda y_\ell.r_\ell$ has been singled out.

NB There is always (at least one) matching case, because intro/elim rules comes from different lines in the truth table.
Reductions on terms for permutations

We add the following reduction rules for permutation conversions.

\[(t \cdot_r [p ; \lambda x.q]) \cdot_{r'} [s ; \lambda y.r] \rightarrow_b t \cdot_r [\overline{p} ; \lambda x.(q \cdot_{r'} [\overline{s} ; \lambda y.r])]\]

Here, \(\lambda x.(q \cdot [\overline{s} ; \lambda y.r])\) should be understood as a sequence \(\lambda x_1.q_1, \ldots, \lambda x_m.q_m\) where each \(q_j\) is replaced by \(q_j \cdot_{r'} [\overline{s} ; \lambda y.r]\).
On optimized terms, one can also, in a canonical way, define detour conversion $\rightarrow_a$ and permutation conversion $\rightarrow_b$.

Detour reduction on optimized terms translates to (multi-step) detour reduction on the full terms.

So, strong normalization on optimized terms follows from strong normalization on full terms.

Other well-known rules, like the general elimination rules studied by Schroeder-Heister and Von Plato, can similarly be translated to our full rules.
**Theorem** The reduction $\rightarrow^b$ is strongly normalizing

$$(t \cdot_r [\overline{p} ; \overline{\lambda x.q}]) \cdot_{r'} [\overline{s} ; \overline{\lambda y.r}] \rightarrow^b t \cdot_r [\overline{p} ; \overline{\lambda x.(q \cdot_{r'} [\overline{s} ; \overline{\lambda y.r}])}]$$

**Proof** The measure $\mid - \mid$ decreases with every reduction step.

$$|x| := 1$$
$$|\{\overline{p} ; \overline{\lambda y.q}\}| := \sum |p_i| + \sum |q_j|$$
$$|t \cdot [\overline{s} ; \overline{\lambda y.u}]| := |t|(2 + \sum |s_k| + \sum |u_\ell|)$$
**Theorem** The reduction $\rightarrow_a$ is strongly normalizing.

$$\{\overline{p, p_j}; \overline{\lambda x.q}\} \cdot [\overline{s}; \overline{\lambda y.r, \lambda y_\ell.r_\ell}] \rightarrow_a r_\ell[y_\ell := p_j]$$

(for the $A_j = A_\ell$ case, $p_j : A_j$ and $y_\ell : A_\ell$ with $A_j = A_\ell$)

$$\{\overline{p}; \overline{\lambda x.q, \lambda x_i.q_i}\} \cdot [\overline{s, s_k}; \overline{\lambda y.r}] \rightarrow_a q_i[x_i := s_k] \cdot [\overline{s, s_k}; \overline{\lambda y.r}]$$

(for the $A_i = A_k$ case, $x_i : A_i$ and $s_k : A_k$ with $A_i = A_k$)

**Proof** We adapt the saturated sets method of Tait.

**Corollary** the combination $\rightarrow_{ab}$ is weakly normalizing. Basically: take the $\rightarrow_b$-normal-form and then contract the innermost $\rightarrow_a$-redex of highest rank. (This generalizes the Gandy-Turing WN proof for simple type theory, $\lambda \rightarrow$.)

We have obtained a proof of Strong Normalization for general IPC$_C$.

Rough outline of the proof (generalizing a proof of SN for IPC by Philippe De Groote):

- Define a “double negation” translation from IPC$_C$ formulas to $\lambda \rightarrow$-types.
- Define a reduction preserving “CPS” translation from IPC$_C$ terms to $\lambda \rightarrow$-parallel.
  ($\lambda \rightarrow$ extended with $[M_1, \ldots, M_n] : A$ if $M_i : A$ for $1 \leq i \leq n$.)
- Prove SN for $\lambda \rightarrow$-parallel.
• Types: \( \sigma ::= o \mid (\sigma \rightarrow \sigma) \)

• Terms: \( M ::= x \mid (M M) \mid (\lambda x. M) \mid [M_1, \ldots, M_n] \) \((n > 1)\).

• Typing rules

\[
\begin{align*}
\Gamma \vdash M: A \rightarrow B \quad \Gamma \vdash N: A & \quad \implies \quad \Gamma \vdash MN: B \\
\Gamma, x: A \vdash M: B & \quad \implies \quad \Gamma \vdash \lambda x. M: A \rightarrow B \\
(x : A) \in \Gamma & \quad \implies \quad \Gamma \vdash x: A \\
\Gamma \vdash M_1: A \quad \ldots \quad \Gamma \vdash M_n: A & \quad \implies \quad \Gamma \vdash [M_1, \ldots, M_n]: A
\end{align*}
\]

• Reduction rules: \((\lambda x. M) N \rightarrow_\beta M[x := N]\) plus

\[
[M_1, \ldots, M_n] N \rightarrow_\gamma [M_1 N, \ldots, M_n N]
\]

SN can be proved by adapting the well-known Tait proof.
Translating formulas to types (outline)

Abbreviate \( \neg A := A \to o \).

- For a proposition letter, \( \hat{A} := \neg\neg A \).
- For \( \Phi = c(A_1, \ldots, A_n) \) with elimination rules \( r_1, \ldots, r_t \)

\[
\hat{\Phi} := \neg (E_1 \to \cdots \to E_t \to o),
\]

where

\[
E_s := \hat{A}_{k_1} \to \cdots \to \hat{A}_{k_m} \to \neg \hat{A}_{l_1} \to \cdots \to \neg \hat{A}_{l_{n-m}} \to o
\]

with the \( A_k \) the 1-entries and the \( A_l \) are the 0-entries in the truth table.

For example

\[
\hat{A} \land B = \neg (\neg \neg \hat{A} \to \neg \neg \hat{B} \to o)
\]

\[
\hat{A} \lor B = \neg ((\neg \hat{A} \to \neg \hat{B} \to o) \to o)
\]
We have a translation \( \hat{M} \) and a second translation \( \hat{\hat{M}} \). (This is a generalization of the CPS translation \( \overline{M} \) of Plotkin, that De Groote also uses.)

We can prove

- If \( M \xrightarrow{\beta} N \), then \( \hat{M} = \hat{N} \)
- If \( \hat{\hat{M}} \subset K \) (\( \hat{M} \) is a subterm of \( K \)), then

\[
\begin{array}{ccc}
M & \xrightarrow{\beta} & \hat{M} \\
\downarrow a & & \downarrow \hat{\hat{M}} \\
N & \xrightarrow{\beta} & \hat{N}
\end{array}
\begin{array}{ccc}
\subset & K
\end{array}
\begin{array}{ccc}
\exists K'
\end{array}
\]

From this we derive Strong Normalization.
Consequences of Normalization

The set of terms in normal form of IPC\(_C\), NF is characterized by the following inductive definition.

- \(x \in \text{NF}\) for every variable \(x\),
- \(\{\overline{p} ; \overline{\lambda y.q}\} \in \text{NF}\) if all \(p_i\) and \(q_j\) are in NF,
- \(x \cdot [\overline{p} ; \overline{\lambda y.q}] \in \text{NF}\) if all \(p_i\) and \(q_j\) are in NF and \(x\) is a variable.

As corollaries of Normalization we have, for an arbitrary set of connectives:

- subformula property
- consistency of the logic
- decidability of the logic
Classical logic

For classical logic, we have:

\[
\begin{array}{c|c}
A_1 & \ldots & A_n & \Phi \\
\hline
p_1 & \ldots & p_n & 0
\end{array}
\quad \iff \quad \vdash \Phi \ldots \vdash A_j \text{ (if } p_j = 1) \ldots A_i \vdash D \text{ (if } p_i = 0) \ldots \\

\text{classical intro}

\begin{array}{c|c}
A_1 & \ldots & A_n & \Phi \\
\hline
r_1 & \ldots & r_n & 1
\end{array}
\quad \iff \quad \Phi \vdash D \ldots \vdash A_j \text{ (if } r_j = 1) \ldots A_i \vdash D \text{ (if } r_i = 0) \ldots

• If \( p_j = 1 \) (or \( r_j = 1 \)) in \( t_c \), then \( A_j \) occurs as Lemma in the rule
• If \( p_j = 0 \) (or \( r_j = 0 \)) \( t_c \), then \( A_i \) occurs as Casus in the rule

We call \( \vdash \Phi \) (resp. \( \Phi \vdash D \)) the major premise and the other hypotheses of the rule we call the minor premises.
Proof terms for classical logic

t ::= x | (λy : A.t) ⋆_r {t ; λx : A.t} | t ⋅_r [t ; λx : A.t]

where x ranges over variables and r ranges over the rules of all the connectives.
The terms are typed using the following derivation rules.

\[
\begin{array}{c}
\Gamma \vdash x_i : A_i \in \Gamma \\
\Gamma, z : \Phi \vdash t : D \ldots \Gamma \vdash p_i : A_i \ldots \ldots \Gamma, y_j : A_j \vdash q_j : D \ldots
\\
\Gamma \vdash (\lambda z : \Phi.t) \star_r \{\bar{p} ; \lambda y : A.q\} : D \\
\Gamma \vdash t : \Phi \ldots \Gamma \vdash p_k : A_k \ldots \ldots \Gamma, y_\ell : A_\ell \vdash q_\ell : D
\\
\Gamma \vdash t \cdot_r [\bar{p} ; \lambda y : A.q] : D
\end{array}
\]
Reduction for proof terms in classical logic

- First perform permutation reductions.
- Then we perform detour reductions.

This is similar to the constructive case, except for now

- a term is in permutation normal form if all lemmas are variables,
- a detour is an elimination of \( \Phi \) followed by an introduction of \( \Phi \).

NB: in constructive logic, a “detour” is an introduction directly followed by an elimination. Here it is the other way around, and the introduction need not follow the elimination directly.

This is the abstract syntax \( N \) for permutation normal forms:

\[
N ::= x \mid (\lambda y : A.N) \star \{\overline{z} ; \lambda x : A.N\} \mid y \cdot [\overline{z} ; \lambda x : A.N],
\]

where \( x, y, z \) range over variables.
Detours for proof terms in classical logic

A detour is a pattern of the following shape

\[(\lambda x : \Phi \ldots (x \cdot [\overline{v} ; \lambda w : A.s]) \ldots) \star \{\overline{z} ; \lambda y : A.q\}\]

that is, an elimination of \(\Phi = c(A_1, \ldots, A_n)\) followed by an introduction of \(\Phi\), with an arbitrary number of steps in between.

For terms in permutation normal form, detours can be eliminated, obtaining a term in normal form which satisfies the sub-formula property.

Notes to the pattern of a detour:
- the indicated occurrence need not be the only occurrence of \(x\)
- variable \(x\) may not occur at all; that is the simplest situation.
Eliminating detours is done by the following reduction steps:

1. \((\lambda x : \Phi \ldots (x \cdot [\overline{v} \; ; \; \overline{\lambda w : A.s}] \ldots) \star \{\overline{z} ; \; \overline{\lambda y : A.q}\} \longrightarrow_a (\lambda x : \Phi \ldots (s_\ell[w_\ell := z_i]) \ldots) \star \{\overline{z} ; \; \overline{\lambda y : A.q}\})\)
   if \(i = \ell (A_i = A_\ell)\) is a “matching case” for the subformulas of \(\Phi\).

2. \((\lambda x : \Phi \ldots (x \cdot [\overline{v} \; ; \; \overline{\lambda w : A.s}] \ldots) \star \{\overline{z} ; \; \overline{\lambda y : A.q}\} \longrightarrow_a (\lambda x : \Phi \ldots (q_j[y_j := v_k]) \ldots) \star \{\overline{z} ; \; \overline{\lambda y : A.q}\})\)
   if \(j = k (A_j = A_k)\) is a “matching case” for the subformulas of \(\Phi\).

3. \((\lambda x : \Phi.t) \star \{\overline{z} ; \; \overline{\lambda y : A.q}\} \longrightarrow_a t\) if \(x \notin \text{FV}(t)\).

Tonny Hurkens has given a proof that this normalizes...
Conclusions

• Simple general way to derive constructive and classical deduction rules for (new) connectives.
• Study connectives “in isolation”. (Without other connectives.)
• Generic Kripke semantics for constructive logic
• General definitions of detour conversion and permutation conversion.
• General Curry-Howard proofs-as-terms interpretation.
• General Strong Normalization proof.
Future work and Related

- Meaning of the new connectives as inductive data types.
- Study conditions for the set of rules to be Church-Rosser.
- Study the computational meaning of classical proof terms.
- Relation with other well-known term calculi for classical logic: subtraction logic (Crolard), $\lambda\mu$ (Parigot), $\bar{\lambda}\mu\bar{\mu}$ (Curien, Herbelin).

Related work:

- Dyckhoff; Milne; von Plato and Negri; Schroeder-Heister; Joachimski and Matthes; Baaz, Fermüller and Zach; Abel; …
- “Harmony” in logic (following Prawitz)
Questions?