Efficient compression of SIDH public keys

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Supersingular-isogeny Diffie-Hellman

- Post-quantum secure (ephemeral) key exchange [JF11]
- Based on hardness of finding large-degree isogenies
- Small keys ($\approx$ 564 bytes public)
- Relatively slow compared to other PQ proposals
- Key compression ($\approx$ 385 bytes), at very high cost [Aza+16]

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- Key compression (≈ 385 bytes), at very high cost [Aza+16]

This talk

- Key size reduced by 12.5% (≈ 330 bytes)
- Compression up to $66\times$ faster
- Decompression up to $15\times$ faster

Isogeny graphs

$$\boxed{p = 2^3 \cdot 3^2 - 1, \quad E/\mathbb{F}_{p^2} : y^2 = x^3 + x, \quad j(E) = 24, \quad \ell = 2}$$

41

$\boxed{24}$

66

17

0

48

40

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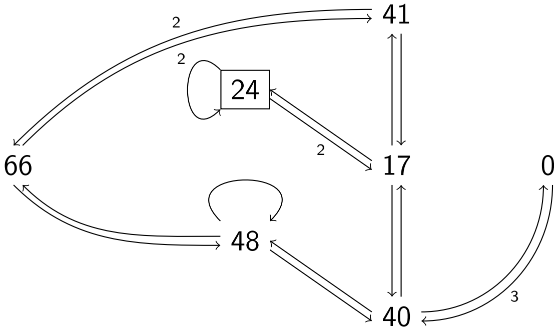

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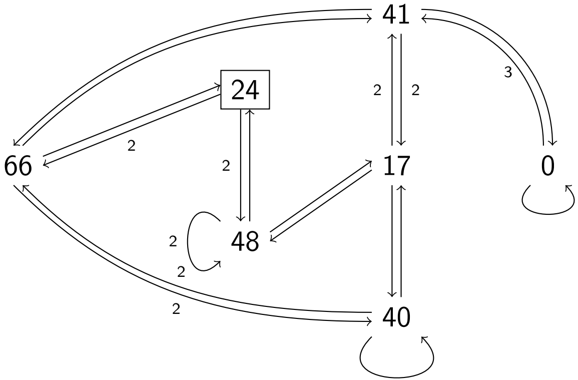
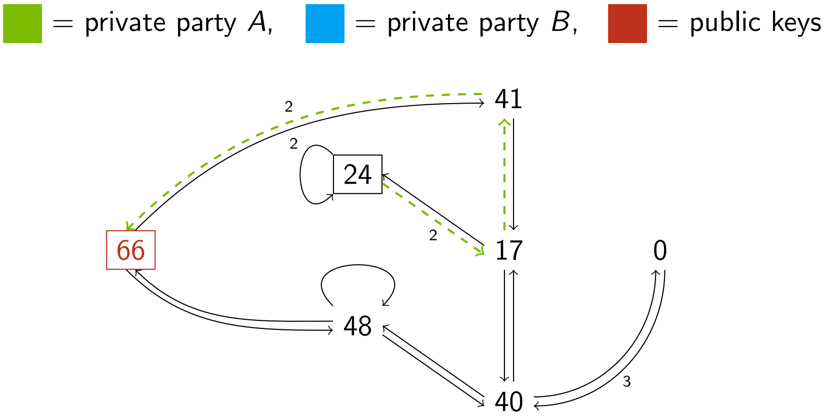
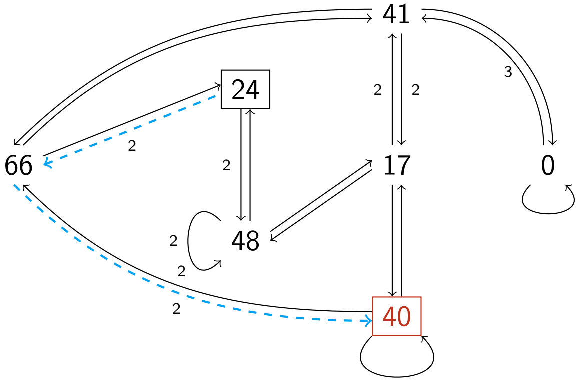


Key generation



Key generation

= private party A, = private party B, = public keys



Supersingular-isogeny Diffie-Hellman [JF11]

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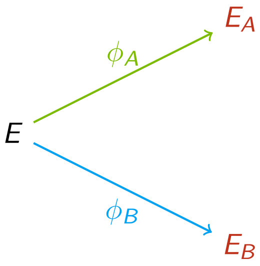

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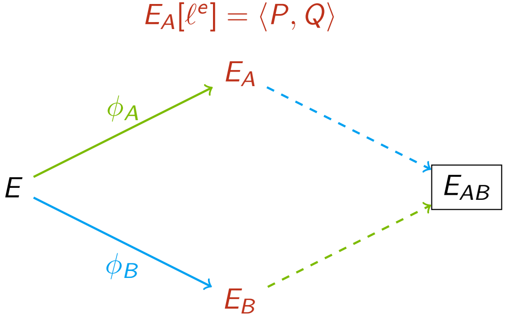

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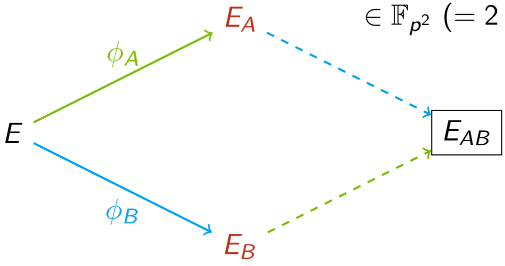

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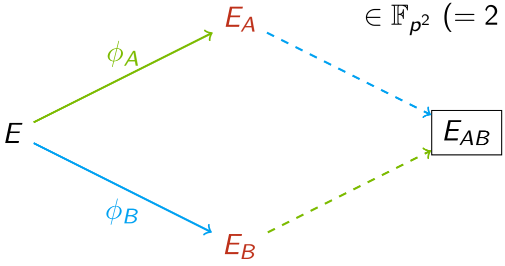

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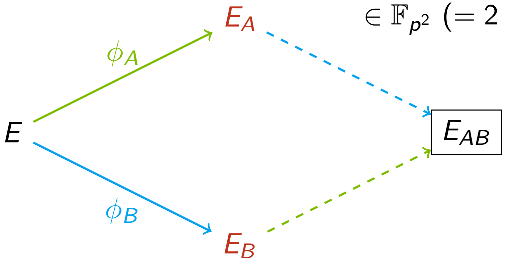
$(\alpha, \beta, \gamma, \delta) \in \mathbb{Z}_{\ell^e}^4$ ($\approx 2\log p$ bits)

$E_A \in \mathbb{F}_{p^2}$ ($= 2\log p$ bits)



Public-key compression [Aza+16]

Compression

$$\langle P, Q\rangle \longrightarrow \begin{matrix}\langle R, S\rangle \\ \langle \alpha R + \beta S, \gamma R + \delta S\rangle\end{matrix} \longrightarrow (\alpha, \beta, \gamma, \delta)$$

Decompression

$$(\alpha, \beta, \gamma, \delta) \longrightarrow \begin{matrix}\langle R, S\rangle \\ (\alpha, \beta, \gamma, \delta)\end{matrix} \longrightarrow \langle P, Q\rangle$$

Public-key compression [Aza+16]

Compression

$$\langle P, Q\rangle \longrightarrow \begin{matrix}\langle R, S\rangle \\ \langle \alpha R + \beta S, \gamma R + \delta S\rangle\end{matrix} \xrightarrow{\text{Expensive}} (\alpha, \beta, \gamma, \delta)$$

Decompression

$$(\alpha, \beta, \gamma, \delta) \longrightarrow \begin{matrix}\langle R, S\rangle \\ (\alpha, \beta, \gamma, \delta)\end{matrix} \longrightarrow \langle P, Q\rangle$$

Public-key compression [Aza+16]

Significantly improve efficiency (up to 66×)

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Significantly improve efficiency (up to 15×)

Finding a canonical basis

Find R, S such that $E[2^{372}] = \langle R, S \rangle$, where

$$\#E(\mathbb{F}_{p^2}) = \left(2^{372} 3^{239}\right)^2 .$$

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Finding an element of order 2^{372}

1. Deterministically pick $R \in E(\mathbb{F}_{p^2}) \setminus 2E(\mathbb{F}_{p^2})$

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For $E : y^2 = x(x-\gamma)(x-\delta)$,

$$R \in 2E(\mathbb{F}_{p^2}) \iff x_R, x_R - \delta, x_R - \gamma \text{ are squares}$$

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4. Set $R \leftarrow [3^{239}]R$

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Finding a canonical basis of $E[2^{372}]$

1. Pick $R \in E(\mathbb{F}_{p^2})$ of order 2^{372}
2. Pick $S \in E(\mathbb{F}_{p^2})$ of order 2^{372}
3. If $E[2^{372}] \neq \langle R, S \rangle$, goto 2.

Transferring to μ_n via reduced Tate pairing

Transfer the discrete logs to μ_n

$$e = e(R, S) \qquad e^{\beta} = e(R, P) \qquad e^{\delta} = e(R, Q)$$
$$e^{-\alpha} = e(S, P) \qquad e^{-\gamma} = e(S, Q)$$

such that $P = \alpha R + \beta S$ and $Q = \gamma R + \delta S$

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- -

$f_0 \leftarrow f_{n,R}$

$f_0 \leftarrow f_0(S)$

$\vdots$

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$e(R,S)$	$e(R,P)$	$e(R,Q)$	$e(S,P)$	$e(S,Q)$
$f_0 \leftarrow f_{n,R}$			$f_3 \leftarrow f_{n,S}$	
$f_0 \leftarrow f_0(S)$	$f_1 \leftarrow f_0(P)$	$f_2 \leftarrow f_0(Q)$	$f_3 \leftarrow f_3(P)$	
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Optimized formulas for $f_{n,R}$ and $f_{n,S}$!

Efficient discrete logarithms (Pohlig-Hellman)

For $e_0, e_1, e_2, e_3, e_4 \in \mu_{\ell^e}$, compute $\alpha, \beta, \gamma, \delta$ such that

$$e_1 = e_0^{-\alpha}, \qquad e_2 = e_0^{\beta}, \qquad e_3 = e_0^{-\gamma}, \qquad e_4 = e_0^{\delta}$$

As $\mu_{\ell^e} \subset G_{p+1} \subset \mathbb{F}_{p^2}$, $\quad \mathbf{I} \approx \mathbf{M}, \quad \mathbf{S} \approx 2\mathbf{s}, \quad \mathbf{C} \approx 2\mathbf{m} + 1\mathbf{s}$

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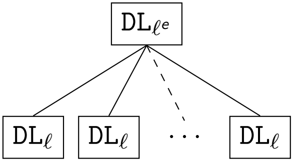
$\#G_1 = \ell^e$

$\#G_1 = \ell$



Nested Pohlig-Hellman

$\#G_1 = \ell^{e_1}$

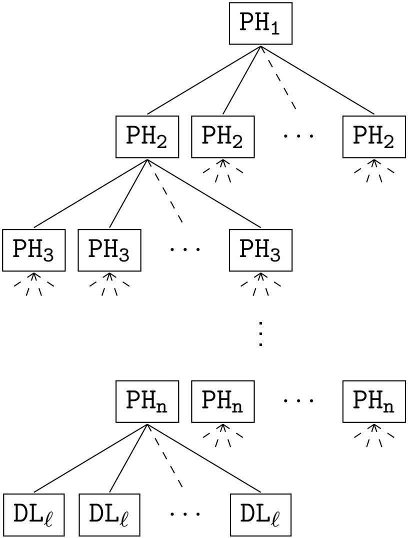
$\#G_2 = \ell^{e_2}$

$\#G_3 = \ell^{e_3}$

$\vdots$

$\#G_n = \ell^{e_n}$

$\#G_{n+1} = \ell$



Comparison

#	windows				$\mathbb{F}_{p^2}$		table size
n	w_1	w_2	w_3	w_4	**M**	**S**	$\mathbb{F}_{p^2}$
0	–	–	–	–	372	69 378	375
1	19	–	–	–	375	7 445	43
2	51	7	–	–	643	4 437	25
3	84	21	5	–	716	3 826	25
4	114	35	11	3	1 065	3 917	27

Options for different time-memory trade-offs [Sut11]

Signature size reduction

- The quadruple $(\alpha, \beta, \gamma, \delta) \in \mathbb{Z}_{\ell^e}^4$ determines

$$P = \alpha R + \beta S, \quad Q = \gamma R + \delta S.$$

  These determine $\langle P + \lambda Q \rangle$, for some $\lambda \in \mathbb{Z}_{\ell^e}^*$
- Thus we only need P, Q up to scalar, and compress to

$$[\alpha : \beta : \gamma : \delta] .$$

  As P, Q form a basis of $E[\ell^e]$, either α or β is invertible
- Normalizing, we represent it in $\mathbb{Z}_{\ell^e}^3 \times \mathbb{Z}_2$

Benchmarks (for $\ell = 2$)

	This work	[Aza+16]	Speed-up
Key size (bytes)	328	385	–
SIDH (cc $\times$ 10^6)	80	–	–
Compression (cc $\times$ 10^6)	109	6 081	56$\times$
Decompression (cc $\times$ 10^6)	42	539	13$\times$
Full no comp. (cc $\times$ 10^6)	192	535	2.8$\times$
Full comp. (cc $\times$ 10^6)	469	15 395	31$\times$

Software available at

https://github.com/Microsoft/PQCrypto-SIDH

Thanks!

Questions?

References I

[Aza+16] Reza Azarderakhsh, David Jao, Kassem Kalach, Brian Koziel and Christopher Leonardi. "Key Compression for Isogeny-Based Cryptosystems". In: *Proceedings of the 3rd ACM International Workshop on ASIA Public-Key Cryptography, AsiaPKC@AsiaCCS, Xi'an, China, May 30 - June 03, 2016*. Ed. by Keita Emura, Goichiro Hanaoka and Rui Zhang. ACM, 2016, pp. 1–10. DOI: 10.1145/2898420.2898421. URL: http://doi.acm.org/10.1145/2898420.2898421.

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