Clustering Approach for Higher-Order Deterministic Masking

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Side-Channel Attacks

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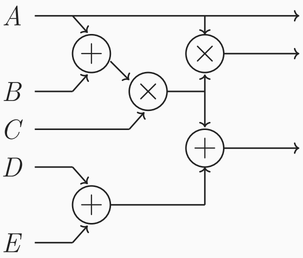

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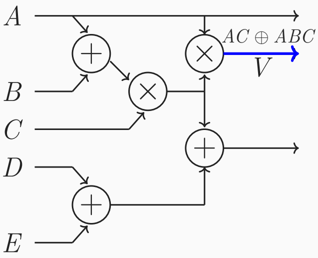

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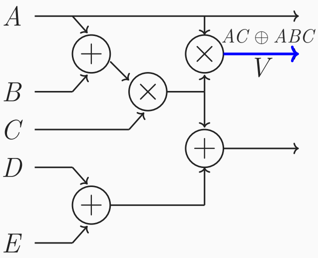

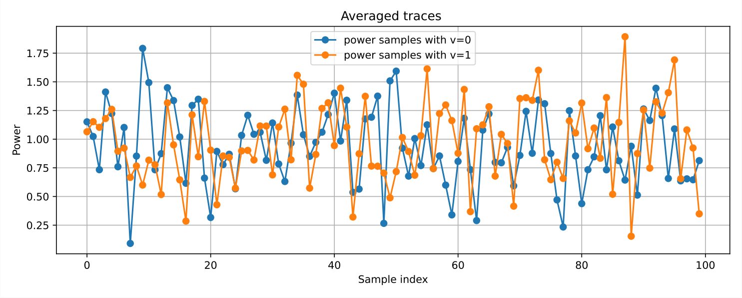

Figure: A single trace.

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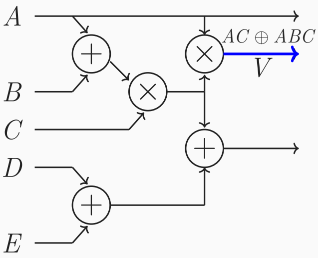

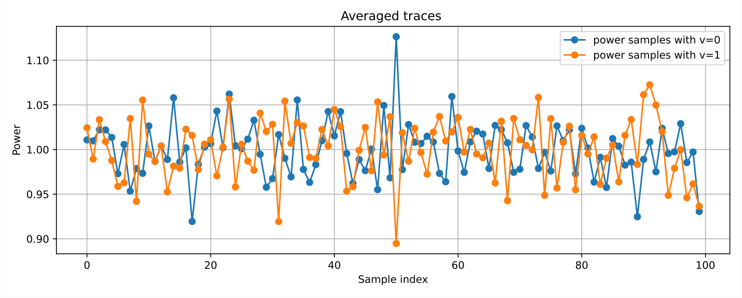

Figure: 100 traces aggregated.

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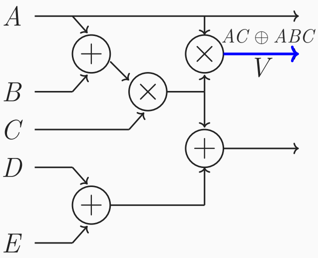

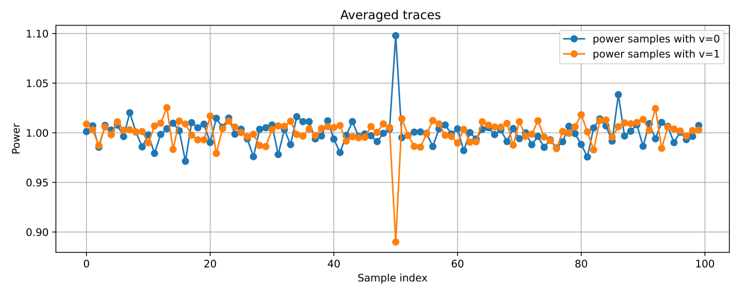

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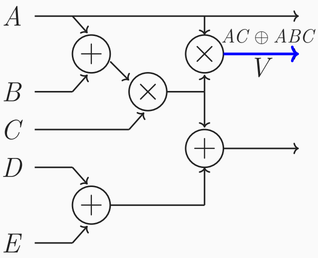

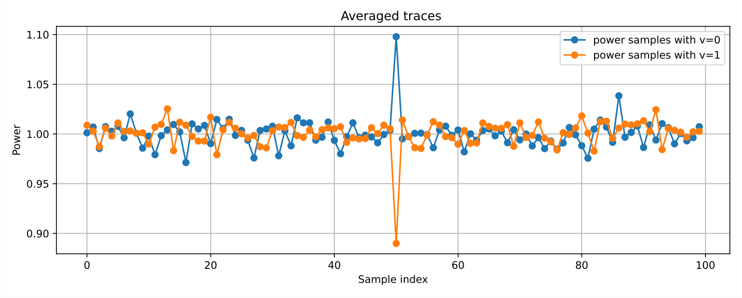

Figure: 1000 traces aggregated.

Take-away: These correlations let an adversary learn operands or even recover secret keys.

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Multiplication (goal)

Given shares $[X_0, \ldots, X_{n-1}]$ and $[Y_0, \ldots, Y_{n-1}]$, securely compute n shares of $Z = XY$ without leaking X or Y.

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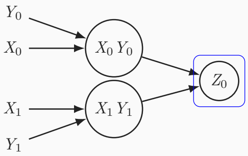

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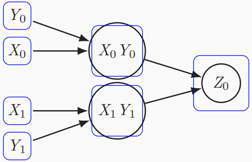

Probe Z_0 (glitch-extended): fan-in exposed.

Example

Let $X = X_0 \oplus X_1$ and $Y = Y_0 \oplus Y_1$. A 2-share "schoolbook" multiplication over $\mathbb{F}_2$:

$$\begin{cases} Z_0 &= X_0\,Y_0 \oplus X_1\,Y_1, \\ Z_1 &= X_0\,Y_1 \oplus X_1\,Y_0. \end{cases}$$

Implication. Probing Z_0 (or Z_1) reveals X and Y.

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Challenge

How can we perform secure multiplication under the glitch-extended probing model without relying on online randomness?

Nikova et al. $(n = 4)$ Deterministic Gadget

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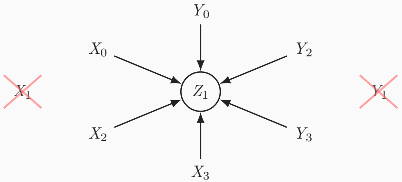

Probe Z_1 (glitch-extended):
fan-in excludes X_1, Y_1.

- **Glitch-extended safety example.** Probing Z_1 does not reveal X_1 or Y_1:
  $\mathrm{fanin}(Z_1) = \{X_0, X_2, X_3, Y_0, Y_2, Y_3\}$.

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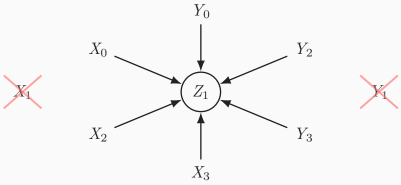

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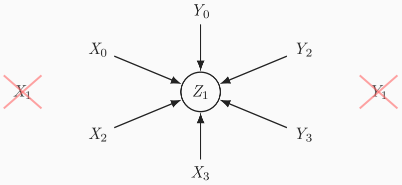

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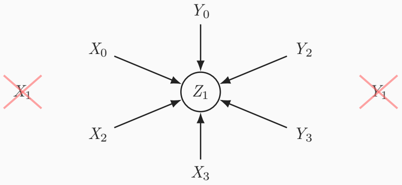

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Computer-Aided Search

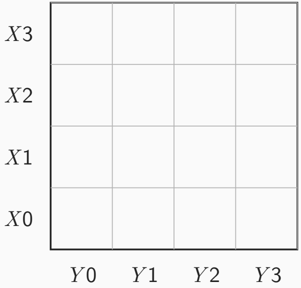
To date, no deterministic multiplication gadget secure against two or more glitch-extended probes is known.

Diving into the Problem (1/3)

Let shares of X be $(X_0, \ldots, X_{n-1})$ and shares of Y be $(Y_0, \ldots, Y_{n-1})$.

$$Z = XY$$

Probe Z_k (glitch-extended) reveals all $(i, j) \in \mathbf{L}_k$



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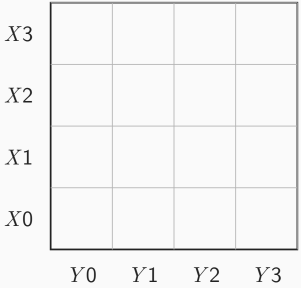

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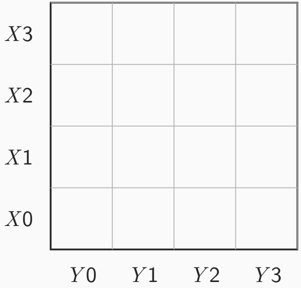

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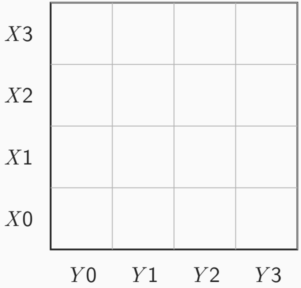

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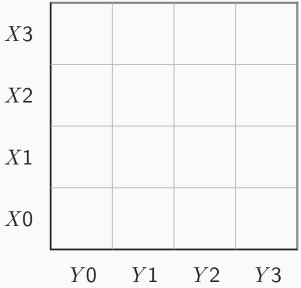

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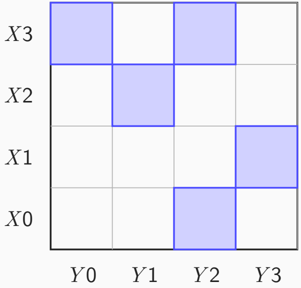

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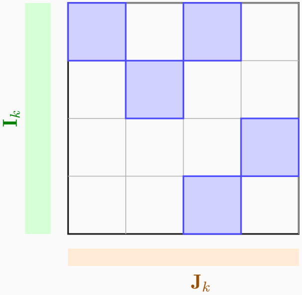

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Goal: choose $\mathbf{L}_k$ so that each probe reveals as few shares as possible.

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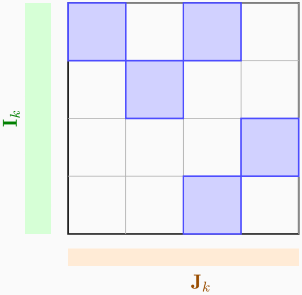

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This gadget is first-order probing secure, but not uniform.

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Can we generalize this pattern for square values of n?

Formalizing the pattern.

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- For $n = s^2$, index the n shares by a grid $\{0, \ldots, s-1\} \times \{0, \ldots, s-1\}$. A **cluster** $C = \{\mathbf{A}_0, \ldots, \mathbf{A}_{s-1}\}$ is a partition of the indices into s multi-shares (blocks) of size s:

$$\mathbf{A}_i \subseteq [n], \quad |\mathbf{A}_i| = s, \quad \mathbf{A}_i \cap \mathbf{A}_j = \varnothing \ (i \neq j), \quad \bigcup_i \mathbf{A}_i = [n].$$

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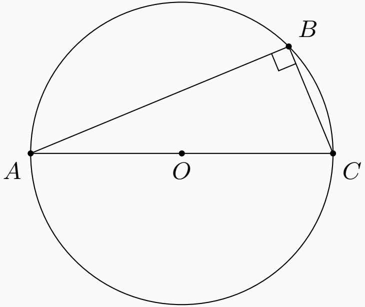
Geometric view. Take $s \times s$ grid points ("shares") as the points of the affine plane. Two parallel classes of lines give exactly such cluster families: lines within a class are parallel (partition), and lines from different classes intersect in one point. This is why our search leads to finite affine geometry.

Euclidean Geometry example



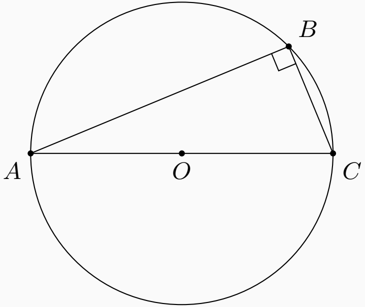


Figure: Thales' Theorem.

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[1] This is actually Playfair's axiom, which is equivalent.

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Trivial examples

Definition (Finite Pre-Affine plane)

A finite pre-affine plane consists of a set of points P and a set of lines L such that it satisfies the postulates (I) and (V).

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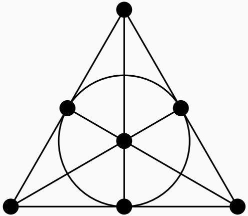

Figure: Fano plane.

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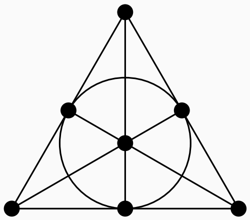

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The Fano plane has 7 points and 7 lines,

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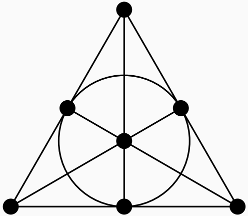

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The Fano plane has 7 points and 7 lines, where each line contains 3 points,

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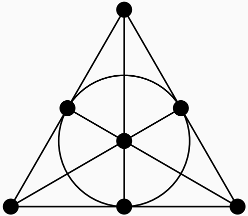

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The Fano plane has 7 points and 7 lines, where each line contains 3 points, each point lies on 3 lines

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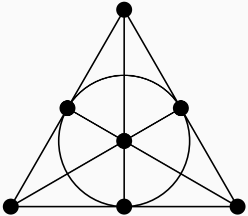

Figure: Fano plane.

The Fano plane has 7 points and 7 lines, where each line contains 3 points, each point lies on 3 lines and all lines intersect.

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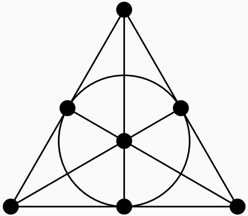

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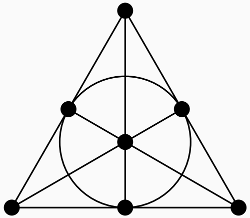

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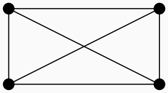

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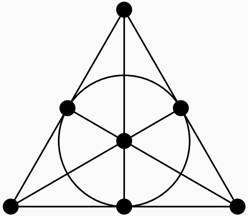

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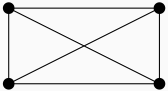
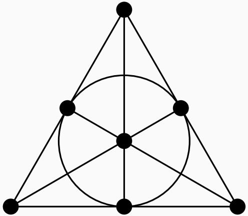


This finite pre-affine plane has 4 points,



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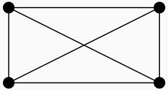

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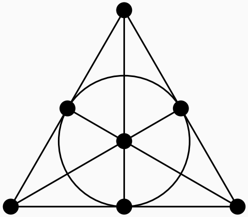

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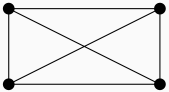

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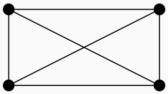

Figure: Affine plane of order 2.

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Figure: P.

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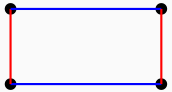

Figure: P and first four lines.

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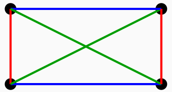

Figure: All points and lines.

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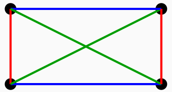

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- We say that the lines ℓ_a have slope ∞.

Example

A finite affine plane of order 4. We take $\mathbb{F} := \mathbb{F}_2[X]/(X^2 + X + 1)$:

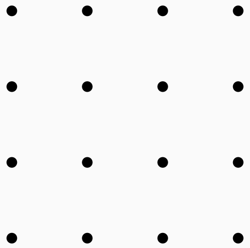

Figure: Points in $\mathbb{F}^2$

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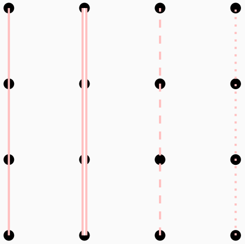

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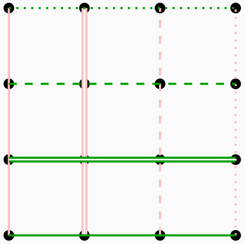

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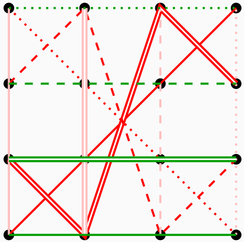

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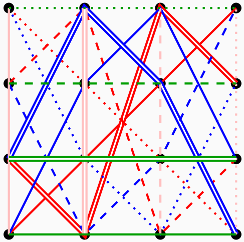

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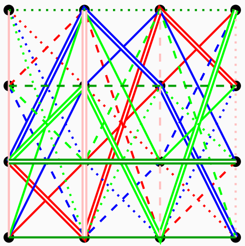

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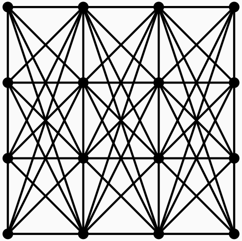

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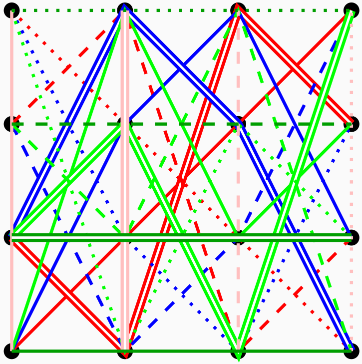

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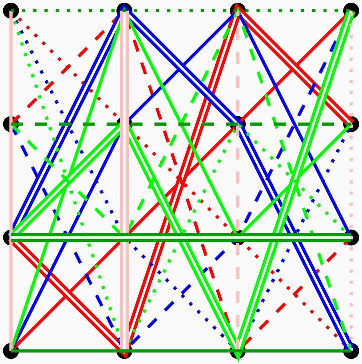

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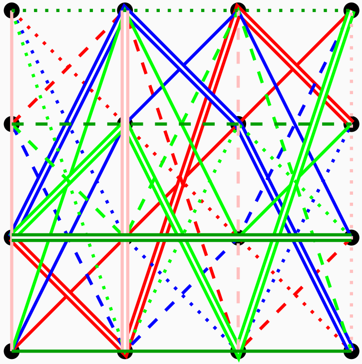

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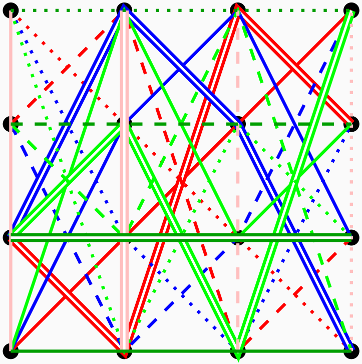
- Known to be possible for all prime powers ($s = p^k$).



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- A cluster that has a missing multi-share can not be completed with less than $s-1$ multi share from other clusters.
- Let s be an odd prime power, and let $n = s^2$. Consider all sums:

$$\bigoplus \mathbf{A}_j^h \quad \text{for} \quad 0 \leq h \leq s, \quad 0 \leq j \leq s-1,$$

where $\mathbf{A}_j^h$ denotes the j-th multi-share from the h-th SMNO cluster of an n-sharing $\mathbf{X}$. Then, the rank of the resulting system of linear (parity) relations is n.

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$$
\begin{aligned}
\alpha &= \lfloor k/s \rfloor, \\
\beta &= \mathrm{mod}(k, s), \\
R_k &= \bigoplus_{i=0}^{s-1} \Big({\mathbf{A}^{0}}_{\alpha}(\mathbf{X})[i] \oplus {\mathbf{A}^{\alpha+1}}_{\beta}(\mathbf{X})[i] \oplus {\mathbf{A}^{0}}_{\alpha}(\mathbf{Y})[i] \oplus {\mathbf{A}^{\alpha+1}}_{\beta}(\mathbf{Y})[i] \Big), \\
W_k &= {\mathbf{A}^{0}}_{\alpha}(\mathbf{X}) \otimes {\mathbf{A}^{\alpha+1}}_{\beta}(\mathbf{Y}), \\
Z_k &= R_k \oplus W_k.
\end{aligned}
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Here, $\mathbf{A}_j^h(\cdot)$ denotes the j-th multi-share from the h-th SMNO cluster applied to the input sharing.

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Thank you for your attention!