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Black holes in algebra: My introduction to Mathieu-Zhao spaces and research

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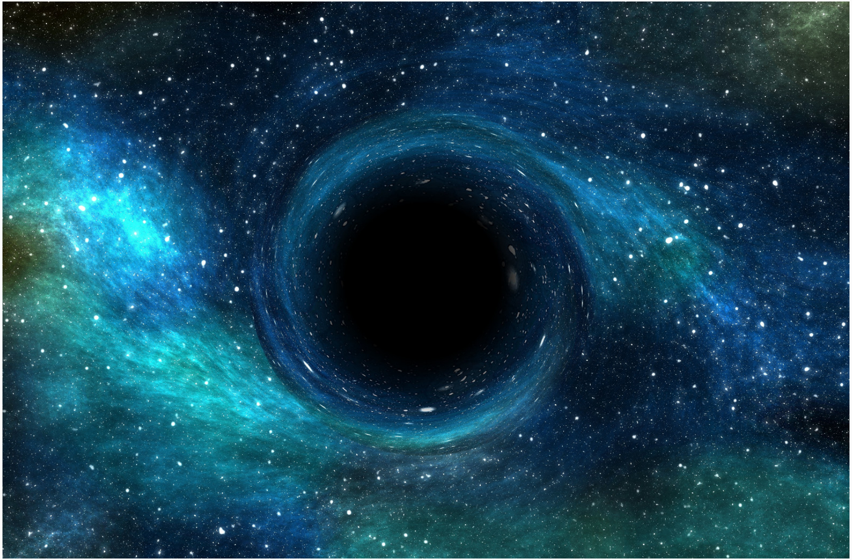

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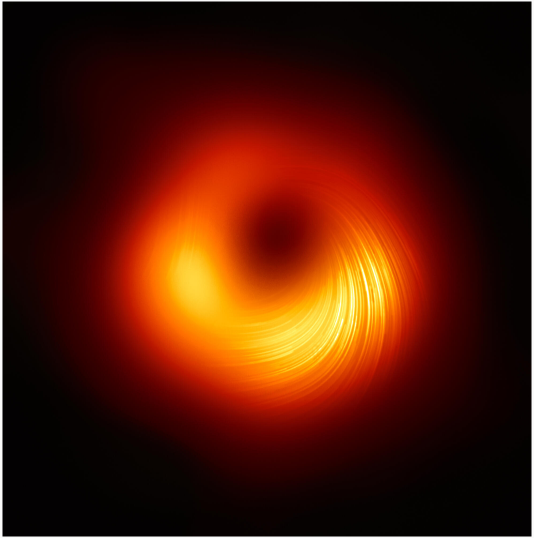

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Consider a quintuple $(R, +, \cdot, 0, 1)$ where R is a set containing special elements that we call 0 and 1, and $+\colon R \times R \to R$ and $\cdot\colon R \times R \to R$ are maps.

1. $\forall a, b, c \in R : (a + b) + c = a + (b + c)$
2. $\forall a \in R : a + 0 = 0 + a = a$
3. $\forall a \in R\ \exists b \in R : a + b = b + a = 0$
4. $\forall a, b \in R : a + b = b + a$
5. $\forall a, b, c \in R : (ab)c = a(bc)$
6. $\forall a \in R : a \cdot 1 = 1 \cdot a = a$
7. $\forall a, b \in R : ab = ba$
8. $\forall a, b, c \in R : a(b + c) = ab + ac$
9. $\forall a, b, c \in R : (b + c)a = ba + ca$

- 1–3: group
- 1–4: abelian group
- 5–6: monoid
- 5–7: commutative monoid
- 1–9: comm. ring

Example of a ring: $(\mathbf{Z}, +, \cdot, 0, 1)$.

Often we just write R and assume the operations are clear, e.g., $\mathbf{Z}$.

$$\left.\begin{array}{l} \text{1. } R \text{ is a commutative ring} \\ \text{2. } \forall a \in R \setminus \{0\}\ \exists b \in R : ab = ba = 1 \end{array}\right\} \text{field}$$

Examples of fields: $\mathbf{Q}, \mathbf{R}, \mathbf{C}$.

Mathieu-Zhao spaces

Recap on ideals

Let R be a ring. An ideal I of R is a(n additive) subgroup of R such that for all a, b in R we have

$$a \in I \implies ba \in I.$$

Hence in particular, for all a, b in R, if for all $m \geq 1$ we have a^m in I, then for all $m \geq 1$ we have ba^m in I:

$$\forall\ m \geq 1\ : a^m \in I \implies \forall\ m \geq 1\ : ba^m \in I$$

Generalising

So for ideals, for all a, b in R:

$$\forall\ m \geq 1\ : a^m \in I \implies \forall\ m \geq 1\ : ba^m \in I.$$

We can relax this a bit, to: for all a, b in R:

$$\forall\ m \geq 1\ : a^m \in I \implies \forall\ m \gg 0\ : ba^m \in I.$$

Or, more commonly written, for all a, b in R:

$$\forall\ m \geq 1\ : a^m \in I \implies \exists\ N > 0\ \forall m \geq N\ : ba^m \in I.$$

Definition

We now define a *Mathieu-Zhao space* of R as a(n additive) subgroup M of R for which the following property holds:

If a^m in M for all $m \geq 1$, then for any b in R we have ba^m in M for all $m \gg 0$.

Example (Ideals)

Ideals of rings.

Not every Mathieu-Zhao space is an ideal!

We now define a *Mathieu-Zhao space* of R as a(n additive) subgroup M of R for which the following property holds:

If a^m in M for all $m \geq 1$, then for any b in R we have ba^m in M for all $m \gg 0$.

Not every Mathieu-Zhao space is an ideal!

Example

Consider the finite field $\mathbb{F}_4 = \{0, 1, x, x+1\}$. We know that $\mathbb{F}_4$ only has two ideals: (0) and $(1) = \mathbb{F}_4$. But the set $M := \{0, x\}$ is also a Mathieu-Zhao space.

We have $x^2 = x + 1$. Since $x + 1$ is not an element of $\{0, x\}$, we find that this set indeed satisfies the conditions for a Mathieu-Zhao space.

Non-example

Let R be any ring. Then $\Delta_R = \{(r,r) \mid r \in R\}$ is a subgroup of $R \times R$, but not a Mathieu-Zhao space:

We have:

$$\forall a \in R \; \forall n \geq 1 : (a,a)^n = (a^n, a^n) \in \Delta_R.$$

Hence, if Δ_R were a Mathieu-Zhao space, then we should have

$$\forall (b,c) \in R \times R \; \exists N \geq 0 \; \forall m \geq N : (b,c)(a,a)^m \in \Delta_R.$$

Let a be any non-nilpotent element ($a^n \neq 0$ for all $n \geq 1$) and $(b,c) = (1,0)$ we have $(1,0)(a,a)^m = (a^m, 0) \notin \Delta_R$ for all $a \neq 0$.

So Δ_R is not a Mathieu-Zhao space.

Background and improvement

Mathieu Conjecture (1995) Let G be a compact connected real Lie group with Haar measure σ. Let f be a complex-valued G-finite function on G such that $\int_G f^m \, d\sigma = 0$ for all $m \geq 1$. Then for every G-finite function g on G, also $\int_G gf^m \, d\sigma = 0$ for all large enough m.

The similarities to Mathieu-Zhao spaces is clear, and we can write (MC) in terms of Mathieu-Zhao spaces:

Mathieu Conjecture Let G be a compact connected real Lie group with Haar measure σ and let A be the algebra of complex-valued G-finite functions on G. Then

$$\left\{ f \in A \mid \int_G f \, d\sigma = 0 \right\}$$

is a Mathieu-Zhao space of A.

Theorem (Duistermaat-Van der Kallen (1998))

Let $X_1, \ldots, X_n$ be n commutative variables and let M be the subspace of the Laurent polynomial algebra $\mathbf{C}[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$ consisting of those Laurent polynomials with no constant term. Then M is a Mathieu-Zhao space of $\mathbf{C}[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$

1-dimensional case:

Theorem (DvdK 1-dimensional)

Let $\mathbf{C}[X, X^{-1}]$ be the Laurent polynomial algebra in one variable. Then

$$\{f \in \mathbf{C}[X, X^{-1}] \mid f_0 = 0\}$$

is a Mathieu-Zhao space of $\mathbf{C}[X, X^{-1}]$.

Duistermaat and Van der Kallen's theorem

The set $\{f \in \mathbf{C}[X, X^{-1}] \mid f_0 = 0\}$ is the kernel of the linear map $L\colon \mathbf{C}[X, X^{-1}] \to \mathbf{C}$ defined by $L(f) = f_0$.

Properties:

- $L(1) \neq 0$;
- $L(X^n) = 0$ for all $n \geq 1$ and all $n \leq -1$.

Theorem (DvdK1 - generalization)

Let $L\colon \mathbf{C}[X, X^{-1}] \to \mathbf{C}$ be a non-zero $\mathbf{C}$-linear map for which there exists an $N \geq 1$ such that $L(X^n) = 0$ for all $n \in \mathbf{Z}_{\geq N}$ and all $n \in \mathbf{Z}_{\leq -N}$. Then $\operatorname{Ker} L$ is a Mathieu-Zhao space of $\mathbf{C}[X, X^{-1}]$ if and only if $L(1) \neq 0$.

From now on we shall say "MZ-space" instead of Mathieu-Zhao space.

Lemma

Let R be a ring. Let M be an MZ-space of R such that $1 \in M$. Then $M = R$.

Proof.

Since $1^m = 1$ for all $m \geq 1$, we find that for all $b \in R$ we have $b1^m \in M$ for all $m \gg 0$ since M is an MZ-space. Hence $b \in M$, and $M = A$. □

Finite rings, e.g., $\mathbf{Z}/100\mathbf{Z}$

Lemma ($\mathbf{Z}/n\mathbf{Z}$)

Let n be a positive integer and let R be the ring $\mathbf{Z}/n\mathbf{Z}$. Then all (additive) subgroups of $\mathbf{Z}/n\mathbf{Z}$ are actually ideals. Since ideals are MZ-spaces, we have now classified all the MZ-spaces of $\mathbf{Z}/n\mathbf{Z}$.

The MZ-spaces of $\mathbf{Z}/100\mathbf{Z}$ are:

$$\langle 0\rangle, \quad \langle 50\rangle, \quad \langle 25\rangle, \quad \langle 20\rangle, \quad \langle 10\rangle, \quad \langle 5\rangle, \quad \langle 4\rangle, \quad \langle 2\rangle, \quad \langle 1\rangle.$$

Alternate notation:

$$0, \quad 50\mathbf{Z}/100\mathbf{Z}, \quad 25\mathbf{Z}/100\mathbf{Z}, \quad 20\mathbf{Z}/100\mathbf{Z},$$

$$10\mathbf{Z}/100\mathbf{Z}, \quad 5\mathbf{Z}/100\mathbf{Z}, \quad 4\mathbf{Z}/100\mathbf{Z}, \quad 2\mathbf{Z}/100\mathbf{Z}, \quad \mathbf{Z}/100\mathbf{Z}.$$

Lemma (Finite Fields)

Let p be a prime, $n \geq 1$ an integer and $q = p^n$. Then all (additive) subgroups of $\mathbb{F}_q$ that do not contain 1 are MZ-spaces of $\mathbb{F}_q$, and of course $\mathbb{F}_q$ itself is also an MZ-space.

Proof.

Let M be a (n additive) subgroup of $\mathbb{F}_q$ that does not contain 1. Let $x \in M$ be such that $x^n \in M$ for all $n \geq 1$. If $x \neq 0$, then this implies $1 \in M$, a contradiction. So only $x = 0$ satisfies the hypothesis $x^n \in M$ for all $n \geq 1$ and clearly for all $y \in \mathbb{F}_q$ we then have $y \cdot 0^m = 0 \in M$ for all $m \gg 0$. △

The finite field $\mathbb{F}_4$ has MZ-spaces $\{0\}, \{0, x\}, \{0, x+1\}, \mathbb{F}_4$.

Main theorems of classification

We introduce here the definition $r(M) = \{a \in A \mid a^n \in M \ \forall n \geq 1\}$. (*We call this the radical of* M.)

Lemma (Radical of nilpotents)

Let R *be a ring and* M *a(n additive) subgroup of* R *with* $r(M) \subset \mathfrak{n}(R)$ *where* $\mathfrak{n}(R)$ *is the set of nilpotent elements of* R*, then* M *is an MZ-space of* R*.*

Theorem (First Classification Theorem)

Let R *be a finite ring. Let* M *be a(n additive) subgroup of* R*. Write* $E(R)$ *for the set of idempotents* ($e^2 = e$) *of* R*. If* $M \cap E(R) = 0$*, then* $r(M) = \mathfrak{n}(R)$ *and* M *is an MZ-space.*

Theorem (Second Classification Theorem')

Let R be a finite ring of the form $R \cong R_1/\mathfrak{m}_1^{k_1} \times R_2/\mathfrak{m}_2^{k_2}$. Then every MZ-space that is not of the form $r(M) \subset \mathfrak{n}(R)$ is of the form $M_1 \times M_2$ where each $M_i \subset R_i/\mathfrak{m}_i^{k_i}$ is an MZ-space of $R_i/\mathfrak{m}_i^{k_i}$.

Example

Let $R := \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$. The product MZ-spaces are:

$$0 = 0 \times 0, \quad 0 \times 2\mathbf{Z}/4\mathbf{Z}, \quad 0 \times \mathbf{Z}/4\mathbf{Z},$$

$$\mathbf{Z}/2\mathbf{Z} \times 0, \quad \mathbf{Z}/2\mathbf{Z} \times 2\mathbf{Z}/4\mathbf{Z}, \quad \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z} = R.$$

By the above theorem, the remaining subspaces have the property that $r(M) \subset \mathfrak{n}(R)$.

Example

Still, $R := \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$. We have previously met the MZ-space $\{(0,0),(1,2)\}$. How do we proceed to find other MZ-spaces M with $r(M) \subset \mathfrak{n}(R)$?

If $M \cap E(R) \neq 0$, then we can determine a non-zero idempotent $e \in M$. Hence $e^n = e \in M$ for all $n \in \mathbf{N}$, and $e \in r(M)$. This contradicts $r(M) \subset \mathfrak{n}(R)$. Thus we must have $M \cap E(R) = 0$.

Furthermore, if there exists some $x \in M$ such that $nx = e$ for some $n \in \mathbf{N}$ and non-zero $e \in E(R)$, then since M is a(n additive) subgroup we have $e \in M$. This contradicts $M \cap E(R) = 0$.

Example

The elements of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$ that are not idempotent or nilpotent are:

$$(0,3), \qquad (1,2), \qquad (1,3).$$

Those nilpotent are

$$(0,0), \qquad (0,2).$$

Note that $3 \cdot (0,3) = (0,1)$ and $3 \cdot (1,3) = (1,1)$.

Leaves: $(0,0), (0,2)$ and $(1,2)$ for M. If $(0,2)$ and $(1,2)$ are both elements of M, then their sum, $(1,0)$ is also an element of M. But this was ruled out before. Hence we have the following possibilities:

$$\{(0,0)\}, \qquad \{(0,0),(0,2)\}, \qquad \{(0,0),(1,2)\}.$$

MZ-spaces of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$

$$0 = 0 \times 0, \qquad 0 \times 2\mathbf{Z}/4\mathbf{Z}, \qquad 0 \times \mathbf{Z}/4\mathbf{Z}, \qquad \mathbf{Z}/2\mathbf{Z} \times 0$$
$$\mathbf{Z}/2Z \times 2\mathbf{Z}/4\mathbf{Z}, \qquad \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z} = R, \qquad \{(0,0),(1,2)\}$$

Thank you all for your attention!